The relational model is injective for Multiplicative Exponential Linear Logic

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Abstract

We prove a completeness result for Multiplicative Exponential Linear Logic (MELL): we show that the relational model is injective for MELL proof-nets, i.e. the equality between MELL proof-nets in the relational model is exactly axiomatized by cut-elimination.

In the seminal paper by Harvey Friedman [11], it has been shown that equality between simply-typed lambda terms in the full typed structure $\mathcal{M}_X$ over an infinte set X is completely axiomatized by β and η: we have $\mathcal{M}_X \vDash v = u \Leftrightarrow v \simeq_{\beta\eta} u$. A natural problem is to know whether a similar result could be obtained for Linear Logic.

Such a result can be seen as a "separation" theorem. To obtain such separation theorems, it is a prerequisite to have a "canonical" syntax. When Jean-Yves Girard introduced Linear Logic (LL) [12], he not only introduced a sequent calculus system but also "proof-nets". Indeed, as for LJ and LK (sequent calculus systems for intuitionnistic and classical logic, respectively), different proofs in LL sequent calculus can represent "morally" the same proof: proof-nets were introduced to find a unique representative for these proofs.

The technology of proof-nets was completely satisfactory for the multiplicative fragment without units.[1] For proof-nets having additives, contractions or weakenings, it was easy to exhibit different proof-nets that should be identified. Despite some flaws, the discovery of proof-nets was striking. In particular, Vincent Danos proved by syntactical means in [3] the confluence of these proof-nets for the Multiplicative Exponential Linear Logic fragment (MELL). For additives, the problem to have a satisfactory notion of proof-net has been addressed in [15]. For MELL, a "new syntax" was introduced in [4]. In the original syntax, the following properties of the weakening and of the contraction did not hold:

- the associativity of the contraction;

- the neutrality of the weakening for the contraction;

- the contraction and the weakening as morphisms of coalgebras.

But they hold in the new syntax; at least for MELL, we got a syntax that was a good candidate to deserve to be considered as being "canonical". Then trying to prove that any two (η-expanded) MELL proof-nets that are equal in some denotational semantics are β-joinable has become sensible and had at least the two following motivations:

- to prove the canonicity of the "new syntax" (if we quotient more normal proof-nets, then we would identify proof-nets having different semantics);

- to prove by semantics means the confluence (if a proof-net reduces to two cut-free proof-nets, then they have the same semantics, so they would be β-joinable, hence equal).

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[1]For the multiplicative fragment with units, it has been recently shown [14] that, in some sense, no satisfactory notion of proof-net can exist. Our proof-nets have no jump, so they identify too many sequent calculus proofs, but not more than the relational semantics.

The problem of *injectivity*[2] of the denotational semantics for MELL, which is the question whether equality in the denotational semantics between (η-expanded) MELL proof-nets is exactly axiomatized by cut-elimination or not, can be seen as a study of the separation property with a semantic approach. The first work on the study of this property in the framework of proof-nets is [16] where the authors deal with the translation into LL of the pure λ-calculus; it has been studied more recently for the intuitionistic multiplicative fragment of LL [17] and for differential nets [18]. For Parigot's $\lambda\mu$-calculus, see [5] and [22].

Finally the precise problem of injectivity for MELL has been adressed by Lorenzo Tortora de Falco in his PhD thesis [23] and in [24] for the (multiset based) coherence semantics and the multiset based relational semantics. He gave partial results and counter-examples for the coherence semantics: the (multiset based) coherence semantics is not injective for MELL. Also, it was conjectured that the relational model is injective for MELL. We prove the conjecture in the present paper.

In [24], a proof of the injectivity of the relational model is given for a weak fragment. But despite many efforts ([23], [24], [1], [19], [18], [20]...), all the attempts to prove the conjecture failed up to now. New progress was made in [9], where it has been proved that the relational semantics is injective for "connected" MELL proof-nets. Still, there, "connected" is understood as a very strong assumption, the set of "connected" MELL proof-nets contains the fragment of MELL defined by removing weakenings and units. Actually [9] proved a much stronger result: in the full MELL fragment two proof-nets R and R' with the same interpretation are the same "up to the connections between the doors of exponential boxes" (we say that they have the same LPS[3] - see Figures 7, 8 and 9 for an example of three different proof-nets having the same LPS). We wrote: "This result can be expressed in terms of differential nets: two cut-free proof-nets with different LPS have different Taylor expansions. We also believe this work is an essential step towards the proof of the full conjecture." Despite the fact we obtained a very interesting result about *all* the proof-nets (i.e. also for non-"connected" proof-nets[4]), the last sentence was a bit too optimistic, since, in the present paper, which presents a proof of the full conjecture, we could not use any previous result nor any previous technic/idea.

The result of the present paper can be seen as

- a semantic separation property in the sense of [11];

- a semantic proof of the confluence property;

- a proof of the "canonicity" of the new syntax of MELL proof-nets;

- a proof of the fact that if the Taylor expansions of two cut-free MELL proof-nets into differential nets [10] coincide, then the two proof-nets coincide.

Let us give one more interpretation of its signifance. First, notice that a proof of this result should consist in showing that, given two non β-equivalent proof-nets R and R', their respective semantics $[\![R]\!]$ and $[\![R']\!]$ are not equal, i.e. $[\![R]\!] \setminus [\![R']\!] \neq \emptyset$ or $[\![R']\!] \setminus [\![R]\!] \neq \emptyset$.[5] But, actually, we prove something much stronger: we prove that, given a proof-net R, there exist two points α and β such that, for any proof-net R', we have $\{\alpha, \beta\} \subseteq [\![R']\!] \Leftrightarrow R \simeq_\beta R'$.

Now, the points of the relational model can be seen as non-idempotent intersection types[6] (see [6] and [7] for a correspondance between points of the relational model and System R - System R has also been studied recently in [2]). And the proof given in the present paper uses the types only to derive the normalization property; actually we prove the injectivity for cut-free proof-nets in an untyped framework:[7] substituting the

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[2]The tradition of the lambda-calculus community rather suggests the word "completeness" and the terminology of category theory rather suggests the word "faithfulness", but we follow here the tradition of the Linear Logic community.

[3]The LPS of a proof-net is the graph obtained by forgetting the outline of the boxes but keeping the auxiliary doors.

[4]and even adding the MIX rule

[5]The converse, i.e. two β-equivalent proof-nets have the same semantics, holds by definition of soundness.

[6]Idempotency of intersection ($\alpha \cap \alpha = \alpha$) does not hold.

[7]For cut-free proof-nets, types guarantee that they are not cyclic as graphs - instead of typing, it is enough to assume this property. Our proof even works for "non-correct" proof-structures (correctness is the property characterizing nets corresponding in a typed framework with proofs in sequent calculus): we could expect that if the injectivity of the relational semantics holds for proof-nets corresponding with MELL sequent calculus, then it still holds for proof-nets corresponding with MELL+MIX sequent calculus, since the category **Rel** of sets and relations is a compact closed category. [13] assuming correctness substituted in the proof the "bridges" of [9] by "empires".

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assumption that proof-nets are typed by the assumption that proof-nets are normalizable does not change anything to the proof.[8] In [8], we gave a semantic characterization of normalizable untyped proof-nets and we characterized "head-normalizable" proof-nets as proof-nets having a non-empty interpretation in the relational semantics. Principal typings in untyped λ-calculus are intersection types which allow to recover all the intersection types of some term. If, for instance, we consider the System R of [6] and [7], it is enough to consider some *injective* 1-*point*[9] to obtain the principal typing of an untyped λ-term. But, generally, for normalizable MELL proof-nets, *injective* k-*points*, for any k, are not principal typings; indeed, two cut-free MELL proof-nets having the same LPS have the same injective k-points for any $k \in \mathbb{N}$. In the current paper we show that a 1-point and a k-*injective point*[10] together allow to recover the interpretation of any normalizable MELL proof-net. So, the result of the current paper can be seen as a first attempt to find a right notion of "principal typing" of intersection types in Linear Logic. As a consequence, normalization by evaluation, as in [21] for λ-calculus, finally becomes possible in Linear Logic too.

Section 1 formalizes PS's (our cut-free proof-nets). Section 2 gives a sketch of our algorithm leading from $[\![R]\!]$ to the rebuilding of R. Section 3 describes more precisely one step of the algorithm and states our theorem (Theorem 9): $[\![R]\!] = [\![R']\!] \Leftrightarrow R \simeq_\beta R'$, where $\simeq_\beta$ is the reflexive symmetric transitive closure of the cut-elimination relation.

Notations. *We denote by ε any empty sequence. If a is a sequence $(\alpha_1, \ldots, \alpha_n)$, then $\alpha_0 : a$ denotes the sequence $(\alpha_0, \ldots, \alpha_n)$; otherwise, it denotes the sequence (α, a) of length 2. The set of finite sequences of elements of some set $\mathcal{E}$ is denoted by $\mathcal{E}^{<\infty}$.*

A multiset f of elements of some set $\mathcal{E}$ is a function $\mathcal{E} \to \mathbb{N}$; we denote by $Supp(f)$ the support of f i.e. the set $\{e \in \mathcal{E}; f(e) \neq 0\}$. A multiset f is said to be finite *if $Supp(f)$ is finite. The set of finite multisets of elements of some set $\mathcal{E}$ is denoted by $\mathcal{M}_{fin}(\mathcal{E})$.*

If f is a function $\mathcal{E} \to \mathcal{E}'$, $x_0 \in \mathcal{E}$ and $y \in \mathcal{E}'$, then we denote by $f[x_0 \mapsto y]$ the function $\mathcal{E} \to \mathcal{E}'$ defined by $f[x_0 \mapsto y](x) = \begin{cases} f(x) & \text{if } x \neq x_0; \\ y & \text{if } x = x_0. \end{cases}$

1 Syntax

We introduce the syntactical objects we are interested in. As recalled in the introduction, simple types guarantee normalization, so we can limit ourselves to nets without any cut. Correctness does not play any role, that is why we do not restrict our nets to be correct and we rather consider proof-structures (*PS's*). Since our proof is easily extended to MELL with axioms, we remove them for simplicity. Moreover, since it is convenient to represent formally our proof using differential nets with boxes (*differential PS's*), we define PS's as differential PS's satisfying some conditions (Definition 3). More generally, *differential $\circ$-PS's* are defined by induction on the depth: Definition 1 concerns what happens at depth 0.

We define the set $\mathbb{T}$ of types as follows: $\mathbb{T} ::= 1 \mid \bot \mid (\mathbb{T} \otimes \mathbb{T}) \mid (\mathbb{T} \,\mathfrak{N}\, \mathbb{T}) \mid !\mathbb{T} \mid ?\mathbb{T}$. We set $\mathfrak{T} = \{\otimes, \mathfrak{N}, 1, \bot, !, ?, \circ\}$. Pre-contractions ($\circ$-ports) are an artefact of our inductive definition on the depth and are used to ensure the canonicity of our syntactical objects (see Example 3).

Definition 1. *A* differential ground-structure *is a 6-tuple $\mathcal{G} = (\mathcal{W}, \mathcal{P}, l, t, \mathcal{L}, \mathcal{T})$, where*

- *$\mathcal{P}$ is a finite set; the elements of $\mathcal{P}(\mathcal{G})$ are the* ports *of $\mathcal{G}$;*

- *l is a function $\mathcal{P} \to \mathfrak{T}$; the element $l(p)$ of $\mathfrak{T}$ is the* label *of p in $\mathcal{G}$;*

- *$\mathcal{W}$ is a subset of $\{p \in \mathcal{P}; l(p) \neq \circ\}$; the elements of $\mathcal{W}(\mathcal{G})$ are the* wires *of $\mathcal{G}$;*

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[8]Except that we have to consider the *atomic* subset of the interpretation instead of the full interpretation (see Remark 6).

[9]An *injective* k-*point* is a point in which all the positive multisets have cardinality k and in which each atom occurring in it occurs exactly twice.

[10]The reader should not confuse k-*injective points* with *injective* k-*points*. k-*injective points* are points in which every positive multiset has cardinality k^j for some $j > 0$ and, for any $j > 0$, there is at most one occurrence of a positive multiset having cardinality k^j - they are obtained by k-*injective experiments* (see our Definition 8).

- t is a function $\mathcal{W} \to \{p \in \mathcal{P}; l(p) \notin \{1, \bot\}\}$ such that, for any port p of $\mathcal{G}$, we have $(l(p) \in \{\otimes, \mathcal{R}\}$ $\Rightarrow \mathsf{Card}(\{w \in \mathcal{W}; t(w) = p\}) = 2)$; if $t(w) = p$, then w is a premise of p; the arity $a_{\mathcal{G}}(p)$ of p is the number of its premises;

- $\mathcal{L}$ is a subset of $\{w \in \mathcal{W}; l(t(w)) \in \{\otimes, \mathcal{R}\}\}$ such that $(\forall p \in \mathcal{P})\, (l(p) \in \{\otimes, \mathcal{R}\}) \Rightarrow \mathsf{Card}(\{w \in \mathcal{L}; t(w) = p\}) = 1)$; if $w \in \mathcal{L}$ s.t. $t(w) = p$, then w is the left premise of p;

- and T is a function $\mathcal{P} \to \mathbb{T}$ such that, for any $p \in \mathcal{P}$,

  - if $l(p) \in \{1, \bot\}$, then $T(p) = l(p)$;
  - if $l(p) = \otimes$ (resp. $l(p) = \mathcal{R}$), then, for any $w_1 \in \mathcal{W} \cap \mathcal{L}$ and any $w_2 \in \mathcal{W} \setminus \mathcal{L}$ such that $t(w_1) = p = t(w_2)$, we have $T(p) = (T(w_1) \otimes T(w_2))$ (resp. $T(p) = (T(w_1) \mathcal{R} T(w_2))$);
  - if $l(p) = !$, then $(\exists C \in \mathbb{T})(T(p) = !C \wedge (\forall w \in \mathcal{W})(t(w) = p \Rightarrow T(w) = C))$;
  - and if $l(p) \in \{\circ, ?\}$, then $(\exists C \in \mathbb{T})(T(p) = ?C \wedge (\forall w \in \mathcal{W})(t(w) = p \Rightarrow T(w) = C))$.

We set $\mathcal{W}(\mathcal{G}) = \mathcal{W}$, $\mathcal{P}(\mathcal{G}) = \mathcal{P}$, $l_{\mathcal{G}} = l$, $t_{\mathcal{G}} = t$, $\mathcal{L}(\mathcal{G}) = \mathcal{L}$, $T_{\mathcal{G}} = T$. The set $\mathcal{P}^{\mathsf{f}}(\mathcal{G}) = \mathcal{P} \setminus \mathcal{W}$ is the set of conclusions of $\mathcal{G}$. For any $t \in \mathfrak{T}$, we set $\mathcal{P}^t(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = t\}$; we set $\mathcal{P}^m(\mathcal{G}) = \mathcal{P}^{\otimes}(\mathcal{G}) \cup \mathcal{P}^{\mathcal{R}}(\mathcal{G})$; the set $\mathcal{P}^e(\mathcal{G})$ of exponential ports of $\mathcal{G}$ is $\mathcal{P}^!(\mathcal{G}) \cup \mathcal{P}^?(\mathcal{G}) \cup \mathcal{P}^{\circ}(\mathcal{G})$.

A ground-structure is a differential ground-structure $\mathcal{G}$ s.t. $\mathsf{im}(t_{\mathcal{G}}) \cap (\mathcal{P}^!(\mathcal{G}) \cup \mathcal{P}^{\circ}(\mathcal{G})) = \emptyset$.

Notice that, for any differential ground-structure $\mathcal{G}$, we have $\mathcal{P}^{\circ}(\mathcal{G}) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G})$.

Example 1. The ground-structure $\mathcal{G}$ defined by: $\mathcal{W}(\mathcal{G}) = \{p_3, p_4, p_5\}$; $\mathcal{P}(\mathcal{G}) = \{p_1, \ldots, p_5\}$; $l_{\mathcal{G}}(p_1) = \bot$, $l_{\mathcal{G}}(p_2) = \mathcal{R}$, $l_{\mathcal{G}}(p_3) = ? = l_{\mathcal{G}}(p_4)$, $l_{\mathcal{G}}(p_5) = 1$; $t_{\mathcal{G}}(p_3) = p_2 = t_{\mathcal{G}}(p_4)$, $t_{\mathcal{G}}(p_5) = p_4$; $\mathcal{L}(\mathcal{G}) = \{p_3\}$; and $T_{\mathcal{G}}(p_1) = \bot$, $T_{\mathcal{G}}(p_2) = (?1 \mathcal{R} ?1)$, $T_{\mathcal{G}}(p_3) = ?1 = T_{\mathcal{G}}(p_4)$, $T_{\mathcal{G}}(p_5) = 1$; is the ground-structure of the content of the box o_1 of R (the leftmost box of Figure 1).

The content of every box of our *differential $\circ$-PS's* is a $\circ$-PS: every !-port inside is always the main door of some box.

Definition 2. For any $d \in \mathbb{N}$, we define, by induction on d, the set of differential $\circ$-PS of depth d (resp. the set of $\circ$-PS of depth d). A differential $\circ$-PS of depth d (resp. a $\circ$-PS of depth d) is a 4-tuple $S = (\mathcal{G}, \mathcal{B}_0, B, b)$, where

- $\mathcal{G}$ is a differential ground-structure (resp. a ground-structure);

- $\mathcal{B}_0 \subseteq \{p \in \mathcal{P}^!(\mathcal{G}); a_{\mathcal{G}}(p) = 0\}$ (resp. $\mathcal{B}_0 = \mathcal{P}^!(\mathcal{G})$) and is the set of boxes of S at depth 0;

- B is a function that associates with every $o \in \mathcal{B}_0$ a $\circ$-PS $B(o) = (\mathcal{G}(B(o)), \mathcal{B}_0(B(o)), B_{B(o)}, b_{B(o)})$ of depth $< d$ that enjoys the following property: if $d > 0$, then there exists $o \in \mathcal{B}_0$ s.t. $B(o)$ is a $\circ$-PS of depth $d - 1$; the !-port o is the main door of the box $B(o)$;

- and b is a function that associates with every $o \in \mathcal{B}_0$ a function $b(o) : \mathcal{P}^{\mathsf{f}}(\mathcal{G}(B(o))) \to \{o\} \cup \mathcal{P}^?(\mathcal{G}) \cup \mathcal{P}^{\circ}(\mathcal{G})$ such that (resp. $\mathcal{P}^{\circ}(\mathcal{G}) \subseteq \bigcup_{o \in \mathcal{B}_0} \mathsf{im}(b(o))$ and), for any $o \in \mathcal{B}_0$,

  - $b(o)|_{\mathcal{P}^{\circ}(\mathcal{G}(B(o)))}$ is injective[11] (resp.[12] $b(o)|_{\mathcal{P}^{\circ}(\mathcal{G}(B(o)))} = id_{\mathcal{P}^{\circ}(\mathcal{G}(B(o)))}$); if $q = b(o)(p)$ with $p \in \mathcal{P}^{\circ}(\mathcal{G}(B(o)))$, then we set $q_{S,o} = p$;
  - $o \in \mathsf{im}(b(o))$ and $\mathsf{im}(b(o)|_{\mathcal{P}^{\circ}(\mathcal{G}(B(o)))}) \cap \mathcal{P}^!(\mathcal{G}) = \emptyset$;
  - for any $p \in \mathsf{dom}(b(o)) \cap \mathcal{P}^{\circ}(\mathcal{G}(B_R(o)))$, we have $T_{\mathcal{G}}(b(o)(p)) = T_{\mathcal{G}(B(o))}(p)$;
  - for any $p \in \mathsf{dom}(b(o)) \setminus \mathcal{P}^{\circ}(\mathcal{G}(B_R(o)))$, we have $T_{\mathcal{G}}(b(o)(p)) \in \{? T_{\mathcal{G}(B(o))}(p), ! T_{\mathcal{G}(B(o))}(p)\}$;

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[11]So one cannot (pre-)contract several $\circ$-ports of the same box.

[12]This stronger condition on $\circ$-PS's is *ad hoc*, but it allows to lighten the notations.
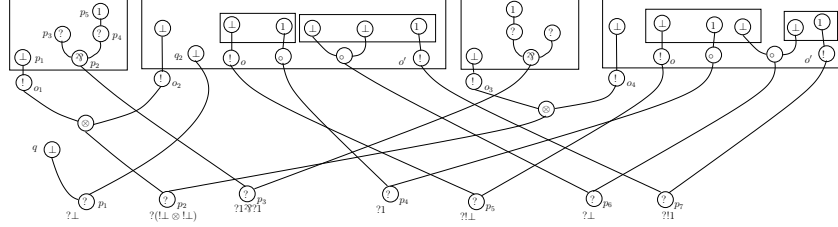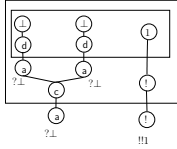
Figure 1: PS R

Figure 2: O_1 ("old syntax")
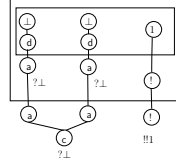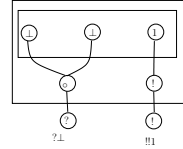

Figure 3: O_2 ("old syntax")

Figure 4: PS N

(resp. moreover no $p \in \mathcal{P}(\mathcal{G})$ is a sequence)[13]. For any differential $\circ$-PS $S = (\mathcal{G}, \mathcal{B}_0, B, b)$, we set $\mathcal{G}(S) = \mathcal{G}$, $\mathcal{B}_0(S) = \mathcal{B}_0$ and $\mathcal{B}(S) = \mathcal{B}_0(S) \cup \bigcup_{o \in \mathcal{B}_0(S)} \{o{:}o'; o' \in \mathcal{B}(B_S(o))\}$ is the set of boxes of S. We denote by B_S the extension of the function B that associates with each $o{:}o' \in \mathcal{B}(S)$, where $o \in \mathcal{B}_0(S)$, the $\circ$-PS $B_{B_S(o)}(o')$. We denote by b_S the extension of the function b that associates with each $o{:}o' \in \mathcal{B}(S)$, where $o \in \mathcal{B}_0(S)$, the function $b_{B_S(o)}(o')$.

We set $\mathcal{W}_0(S) = \mathcal{W}(\mathcal{G}(S))$ and $\mathcal{P}_0(S) = \mathcal{P}(\mathcal{G}(S))$; the elements of $\mathcal{P}_0(S)$ (resp. of $\mathcal{W}_0(S)$) are the ports of S at depth 0 (resp. the wires of S at depth 0). For any $l \in \mathfrak{T} \cup \{m, e\}$, we set $\mathcal{P}_0^l(S) = \mathcal{P}^l(\mathcal{G}(S))$. We set $\mathcal{P}^f(S) = \mathcal{P}^f(\mathcal{G}(S))$, $\mathcal{P}_\circ^f(S) = \mathcal{P}^\circ(\mathcal{G}(S))$ and $\mathcal{P}_\bullet^f(S) = \mathcal{P}^f(S) \setminus \mathcal{P}_\circ^f(S)$; the elements of $\mathcal{P}^f(S)$ are the conclusions of S and the elements of $\mathcal{P}_\circ^f(S)$ are the $\circ$-conclusions of S. For any relation $P \in \{\geq, =, <\}$ on $\mathbb{N}$, for any $i \in \mathbb{N}$, we set $\mathcal{B}_0^{Pi}(S) = \{o \in \mathcal{B}_0(S); depth(B_S(o))Pi\}$ and $\mathcal{B}^{Pi}(S) = \{o \in \mathcal{B}(S); depth(B_S(o))Pi\}$.

PS's are the MELL proof-nets studied in the present paper: there is no cut and no assumption of correctness property.

Definition 3. A PS is a $\circ$-PS R such that $\mathcal{P}_\circ^f(R) = \emptyset$.

Example 2. Consider the PS R of Figure 1. We have $\mathcal{B}_0(R) = \{o_1, o_2, o_3, o_4\}$, $\mathcal{B}(R) = \{o_1, o_2, o_3, o_4, (o_2, o), (o_2, o'), (o_4, o), (o_4, o')\}$, $\mathcal{B}^{=0}(R) = \{o_1, (o_2, o), (o_2, o'), o_3, (o_4, o), (o_4, o')\}$. We have $b_R(o_2)(o) = p_5$, $b_R(o_2)(p_4) = p_4$, $b_R(o_2)(p_6) = p_6$ and $b_R(o_2)(o') = p_7$.

Example 3. In order to understand the role of the $\circ$-ports, consider how the proof-nets O_1 (Figure 2) and O_2 (Figure 3) in the "old syntax" (we denoted derelictions, contractions and auxiliary doors of the "old syntax" by d, c and a, respectively) are represented by the same PS N (Figure 4). Roughly speaking, in our formalism, one pre-contracts (using $\circ$-ports) as soon as possible and one contracts (using ?-ports) as late as possible.

We write $R \simeq R'$ (resp. $R \equiv R'$) if R and R' are the same differential PS's up to the names of their ports (resp. that are not conclusions):

Definition 4. An isomorphism $\varphi : \mathcal{G} \simeq \mathcal{G}'$ of ground-structures is a structure-preserving bijection $\mathcal{P}_0(\mathcal{G}) \simeq \mathcal{P}_0(\mathcal{G}')$. We define, by induction on $depth(R)$, when $\varphi : R \simeq R'$ holds for two differential $\circ$-PS's R and R': it holds whenever φ is a pair $(\varphi_\mathcal{G}, (\varphi_o)_{o \in \mathcal{B}_0(R)})$ s.t. $\varphi_\mathcal{G} : \mathcal{G}(R) \simeq \mathcal{G}(R')$, $\mathcal{B}_0(R') = im(\varphi_\mathcal{G}{}_{|\mathcal{P}_0^!(R)})$ and, for any $o \in \mathcal{B}_0(R)$, $\varphi_o : B_R(o) \simeq B_{R'}(\varphi_\mathcal{G}(o))$ and $(\forall q \in \mathcal{P}^f(B_R(o)))b_{R'}(\varphi_\mathcal{G}(o))(\mathcal{G}(\varphi_o)(q)) = \varphi_\mathcal{G}(b_R(o)(q))$. We set $\mathcal{G}(\varphi) = \varphi_\mathcal{G}$ and, for any $o \in \mathcal{B}_0(R)$, $\varphi(o) = \varphi_o$. We write $\varphi : R \equiv R'$ if $\varphi : R \simeq R'$ s.t. $\mathcal{G}(\varphi)_{|\mathcal{P}^f(R)} = id_{\mathcal{P}^f(R)}$.

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[13]This condition on $\circ$-PS's is *ad hoc*, but it allows to simplify Definition 9.

We write $R \simeq R'$ (resp. $R \equiv R'$) if there exists φ s.t. $\varphi : R \simeq R'$ (resp. $\varphi : R \equiv R'$).

The *arity $a_R(q)$ of a port q in a differential $\circ$-PS R* is computed by "ignoring" the $\circ$-conclusions of the boxes of R:

Definition 5. *Let R be a differential $\circ$-PS. We define, by induction on $depth(R)$, the integers $a_R(q)$ for any $q \in \mathcal{P}_0(R)$ and $cosize(R)$: we set $a_R(q) = a_{\mathcal{G}(R)}(q) + \sum_{o \in \mathcal{B}_0(R)} \mathsf{Card}\left(\{p \in \mathcal{P}^f_\bullet(B_R(o)); b_R(o)(p) = q\}\right) + \sum_{\substack{o \in \mathcal{B}_0(R) \\ q \in im(b_R(o)|_{\mathcal{P}^f_\circ(B_R(o))})}} a_{B_R(o)}(q_{R,o})$ and $cosize(R) = \max(\{a_R(p); p \in \mathcal{P}_0(R)\} \cup \{cosize(B_R(o)); o \in \mathcal{B}_0(R)\}).$*

Example 4. *We have $a_R(p_6) = 4$ (and not 2) and $cosize(R) = 4$ (see Figure 1).*

2 Experiments and their partial expansions

When Jean-Yves Girard introduced proof-nets in [12], he also introduced *experiments of proof-nets*. Experiments (see our Definition 6) are a technology allowing to compute pointwise the interpretation $\llbracket R \rrbracket$ of a proof-net R in the model directly on the proof-net rather than through some sequent calculus proof obtained from one of its sequentializations: the set of *results* of all the experiments of a given proof-net is its interpretation $\llbracket R \rrbracket$. In an untyped framework, experiments correspond with type derivations and results correspond with intersection types.

Definition 6. *For any $C \in \mathbb{T}$, we define, by induction on C, the set $\llbracket C \rrbracket$: $\llbracket 1 \rrbracket = \{*\} = \llbracket \bot \rrbracket$; $\llbracket (C_1 \otimes C_2) \rrbracket = \llbracket C_1 \rrbracket \times \llbracket C_2 \rrbracket = \llbracket (C_1 \parr C_2) \rrbracket$; $\llbracket !C \rrbracket = \mathcal{M}_{fin}(\llbracket C \rrbracket) = \llbracket ?C \rrbracket.$*

Let R be a differential $\circ$-PS. We define, by induction on $depth(R)$, the set of experiments *of R: it is is the set of triples $(R, e_\mathcal{P}, e_\mathcal{B})$, where $e_\mathcal{P}$ is a function that associates with every $p \in \mathcal{P}_0(R)$ an element of $\llbracket T_{\mathcal{G}(R)}(p) \rrbracket$ and $e_\mathcal{B}$ is a function which associates to every $o \in \mathcal{B}_0(R)$ a finite multiset of experiments of $B_R(o)$ such that*

- *for any $p \in \mathcal{P}^m_0(R)$, for any $w_1, w_2 \in \mathcal{W}_0(R)$ such that $t_{\mathcal{G}(R)}(w_1) = p = t_{\mathcal{G}(R)}(w_2)$, $w_1 \in \mathcal{L}(\mathcal{G}(R))$ and $w_2 \notin \mathcal{L}(\mathcal{G}(R))$, we have $e_\mathcal{P}(p) = (e_\mathcal{P}(w_1), e_\mathcal{P}(w_2))$;*

- *for any $p \in \mathcal{P}^e_0(R)$, we have $e(p) = \sum_{\substack{w \in \mathcal{W}_0(R) \\ t_{\mathcal{G}(R)}(w) = p}} [e_\mathcal{P}(w)] + \sum_{o \in \mathcal{B}_0(R)} \sum_{e' \in Supp(e_\mathcal{B}(o))} \left(\sum_{\substack{q \in \mathcal{P}^f_\bullet(B_R(o)) \\ b_R(o)(q) = p}} e_\mathcal{B}(o)(e') \cdot [e'_\mathcal{P}(q)] + \sum_{\substack{q \in \mathcal{P}^f_\circ(B_R(o)) \\ b_R(o)(q) = p}} e_\mathcal{B}(o)(e') \cdot e'_\mathcal{P}(q) \right).$*

For any experiment $e = (R, e_\mathcal{P}, e_\mathcal{B})$, we set $\mathcal{P}(e) = e_\mathcal{P}$ and $\mathcal{B}(e) = e_\mathcal{B}$. We set $\llbracket R \rrbracket = \{\mathcal{P}(e)|_{\mathcal{P}^f(R)}; e$ is an experiment of $R\}$.

We encode in a more compact way the "relevant" information given by an experiment *via pseudo-experiments* and the functions $e^\#$:

Definition 7. *For any differential $\circ$-PS R, we define, by induction on $depth(R)$, the set of* pseudo-experiments *of R: it is the set of functions that associate with every $o \in \mathcal{B}_0(R)$ a finite set of pseudo-experiments of $B_R(o)$ and with ε a pair (R, m) for some $m \in \mathbb{N}$.*

Given an experiment e of some differential $\circ$-PS R, we define, by induction on $depth(R)$, a pseudo-experiment $\overline{e}$ of R as follows: $\overline{e}(\varepsilon) = (R, 1)$ and $\overline{e}(o) = \bigcup_{f \in Supp(\mathcal{B}(e)(o))} \{\overline{f}[\varepsilon \mapsto (B_R(o), i)]; 1 \leq i \leq \mathcal{B}(e)(o)(f)\}$ for any $o \in \mathcal{B}_0(R)$.

Given a pseudo-experiment e of a differential $\circ$-PS R, we define, by induction on $depth(R)$, the function $e^\# : \mathcal{B}(R) \to \mathcal{P}_{fin}(\mathbb{N})$ as follows: for any $o \in \mathcal{B}_0(R)$, $e^\#(o) = \{\mathsf{Card}(e(o))\}$ and, for any $o' \in \mathcal{B}(B_R(o))$, $e^\#(o\text{:}o') = \bigcup_{e' \in e(o)} e'^\#(o').$

There are different kinds of experiments:

- In [24], it was shown that given the result of an *injective k-obsessional experiment* (k big enough) of a cut-free proof-net in the fragment $A ::= X \mid ?A \,\invamp\, A \mid A \,\invamp\, ?A \mid A \otimes A \mid !A$, one can rebuild the entire experiment and, so, the entire proof-net. There, "injective" means that the experiment labels two different axioms with different atoms and "obsessional" means that different copies of the same axiom are labeled by the same atom.

- In [9], it was shown that for any two cut-free MELL proof-nets R and R', we have $LPS(R) = LPS(R')$ iff, for k big enough[14], there exist an *injective k-experiment* of R and an *injective k-experiment* of R' having the same result; as an immediate corollary we obtained the injectivity of the set of (recursively) connected proof-nets. There, "injective" means that not only the experiment labels two different axioms with different atoms, but it labels also different copies of the same axiom by different atoms. Given some proof-net R, there is exactly one injective k-experiment of R up to the names of the atoms.

- In the present paper we show that, for any two PS's R and R', given the result α of a *k-injective experiment* of R for k big enough, if $\alpha \in [\![R']\!]$, then R' is the same PS as R. The conditions on k are given by the result of a 1-experiment, so we show that two (well-chosen) points are enough to determine a PS. The expression "k-injective" means that, for any two different occurrences of boxes, the experiment never takes the same number of copies: it takes k^{j_1} copies and k^{j_2} copies with $j_1 \neq j_2$ (*a contrario*, in [24] and [9], the experiments always take the same number of copies). As shown by the proof-net S of Figure 10, it is impossible to rebuild the experiment from its result, since there exist four different 4-injective experiments e_1, e_2, e_3 and e_4 such that, for any $i \in \{1, 2, 3, 4\}$, we have $e_i(p) = (*, *)$, $e_i(o_1) = [*, *, *, *]$ and $e_i(p') = [[\underbrace{*, \ldots, *}_{4^2}], \ldots, [\underbrace{*, \ldots, *}_{4^6}]]$. For instance e_1 takes 4 copies of the box o_1 and 16 copies of the box o_2, while e_2 takes 4 copies of the box o_1 and 64 copies of the box o_2.

Definition 8. *Let $k > 1$. A pseudo-experiment e of a $\circ$-PS R is said to be k-injective if*

- *for any $o \in \mathcal{B}(R)$, for any $m \in e^{\#}(o)$, there exists $j > 0$ such that $m = k^j$;*

- *for any $o \in \mathcal{B}_0(R)$, for any $o' \in \mathcal{B}(B_R(o))$, we have $(\forall e_1, e_2 \in e(o))\ (e_1{}^{\#}(o') \cap e_2{}^{\#}(o') \neq \emptyset \Rightarrow e_1 = e_2)$;*

- *and, for any $o_1, o_2 \in \mathcal{B}(R)$, we have $(e^{\#}(o_1) \cap e^{\#}(o_2) \neq \emptyset \Rightarrow o_1 = o_2)$.*

An experiment e is said to be k-injective if $\overline{e}$ is k-injective.

Example 5. *There exists a 10-injective pseudo-experiment f of the proof-net R of Figure 1 such that $f^{\#}(o_1) = \{10^{223}\}$, $f^{\#}(o_2) = \{10\}$, $f^{\#}(o_3) = \{10^{224}\}$, $f^{\#}(o_4) = \{100\}$, $f^{\#}((o_2, o)) = \{10^3, \ldots, 10^{12}\}$, $f^{\#}((o_2, o')) = \{10^{13}, \ldots, 10^{22}\}$, $f^{\#}((o_4, o)) = \{10^{23}, \ldots, 10^{122}\}$ and $f^{\#}((o_4, o')) = \{10^{123}, \ldots, 10^{222}\}$.*

In [9], the interest for *injective* experiments came from the remark that the result of an *injective* experiment of a *cut-free* proof-net can be easily identified with a differential net of its Taylor expansion in a sum of differential nets [10] (it is essentialy the content of our Lemma 1). Thus any proof using injective experiments can be straightforwardly expressed in terms of differential nets and conversely. Since this identification is trivial, besides the idea of considering injective experiments instead of obsessional experiments, the use of the terminology of differential nets does not bring any new insight[15], it just superficially changes the presentation.That is why we decided in [9] to avoid introducing explicitly differential nets. In the present paper, we made the opposite choice for the following reason: the algorithm leading from the result of a k-injective experiment of R to the entire rebuilding of R is done in several steps: in the intermediate steps, we obtain a partial rebuilding where some boxes have been recovered but not all of them; a convenient way to represent

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[14]Interestingly, [13], following the approach of [9], showed that, if these two proof-nets are assumed to be (recursively) connected, then we can take $k = 2$.

[15]For proof-nets with cuts, the situation is completely different: the great novelty of differential nets is that differential nets have a cut-elimination; the differential nets appearing in the Taylor expansion of a proof-net with cuts have cuts, while the semantics does not see these cuts. But the proofs of the injectivity only consider cut-free proof-nets.
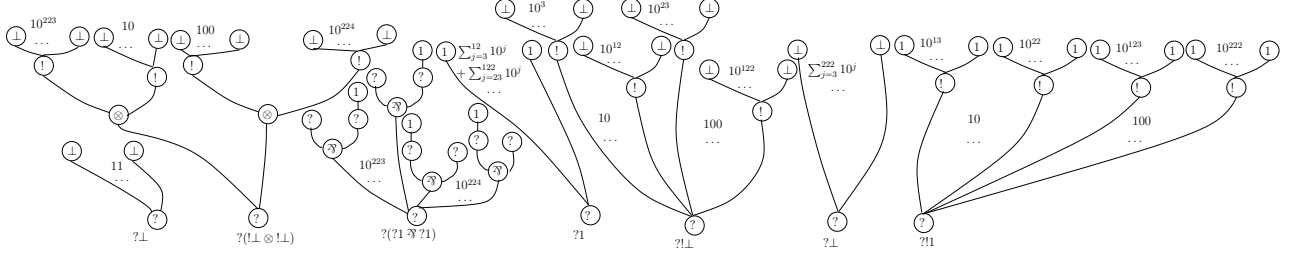
Figure 5: $\mathcal{T}(f)[0]$

this information is the use of "differential nets with boxes" (called "differential PS's" in the present paper). Now, the differential net representing the result and the proof-net R are both instances of the more general notion of "differential nets with boxes".

The rebuilding of the proof-net R is done in d steps, where d is the depth of R. We first rebuild the occurrences of the boxes of depth 0 (the deepest ones) and next we rebuild the occurrences of the boxes of depth 1 and so on... This can be formalized using differential nets (with boxes) as follows: if e is an injective experiment of R, then $\mathcal{T}(\overline{e})[i]$ is the differential net corresponding with e in which only boxes of depth $\geq i$ are expanded,[16] so $\mathcal{T}(\overline{e})[0]$ is (essentially) the same as the result of the experiment and $\mathcal{T}(\overline{e})[d] = R$; the first step of the algorithm builds $\mathcal{T}(\overline{e})[1]$ from $\mathcal{T}(\overline{e})[0]$, the second step builds $\mathcal{T}(\overline{e})[2]$ from $\mathcal{T}(\overline{e})[1]$, and so on... We thus reduced the problem of the injectivity to the problem of rebuilding $\mathcal{T}(\overline{e})[i+1]$ from $\mathcal{T}(\overline{e})[i]$ for any k-injective experiment e (k big enough).

Definition 9. *Let R be a $\circ$-PS of depth d. Let e be a pseudo-experiment of R. Let $i \in \mathbb{N}$. We define, by induction on d, a differential $\circ$-PS $\mathcal{T}(e)[i] = ((\mathcal{W}_{e,i}, \mathcal{P}_{e,i}, l_{e,i}, t_{e,i}, \mathcal{L}_{e,i}), \mathcal{B}_{e,i}, B_{e,i}, b_{e,i})$ of depth $\min\{i,d\}$ s.t. $\mathcal{P}^f(R) = \mathcal{P}^f(\mathcal{T}(e)[i])$ and $(\forall p \in \mathcal{P}^f(R)) l_{\mathcal{G}(R)}(p) = l_{\mathcal{G}(\mathcal{T}(e)[i])}(p)$ as follows: we set $\mathcal{P}^{\bullet}_{e,i} = \bigcup_{o_1 \in \mathcal{B}^{\geq i}_0(R)} \bigcup_{e_1 \in e(o_1)} \big\{(o_1,e_1):p; p \in \mathcal{P}_{e_1,i} \setminus \mathcal{P}^f_{\circ}(B_R(o_1))\big\}$;*

- $\mathcal{W}_{e,i} = \mathcal{W}_0(R) \cup \mathcal{P}^{\bullet}_{e,i}$ *and* $\mathcal{P}_{e,i} = \mathcal{P}_0(R) \cup \mathcal{P}^{\bullet}_{e,i}$;

- $l_{e,i}(p) = \left\{ \begin{array}{ll} l_{\mathcal{G}(R)}(p) & \text{if } p \in \mathcal{P}_0(R); \\ l_{e_1,i}(p') & \text{if } p = (o_1,e_1):p' \text{ with } o_1 \in \mathcal{B}^{\geq i}_0(R); \end{array} \right.$

- $t_{e,i}$ *is the extension of* $t_{\mathcal{G}(R)}$ *that associates with each* $(o_1,e_1):w' \in \mathcal{W}_{e,i}$*, where* $o_1 \in \mathcal{B}^{\geq i}_0(R)$*, the port*
$\left\{ \begin{array}{ll} (o_1,e_1):t_{e_1,i}(w') & \text{if } w' \in \mathcal{W}_{e_1,i} \text{ and } t_{e_1,i}(w') \notin \mathcal{P}^f_{\circ}(B_R(o_1)); \\ b_R(o_1)(t_{e_1,i}(w')) & \text{if } w' \in \mathcal{W}_{e_1,i} \text{ and } t_{e_1,i}(w') \in \mathcal{P}^f_{\circ}(B_R(o_1)); \\ b_R(o_1)(w') & \text{if } w' \in \mathcal{P}^f_{\bullet}(B_R(o_1)); \end{array} \right.$

- $\mathcal{L}_{e,i} = \mathcal{L}(\mathcal{G}(R)) \cup \bigcup_{o_1 \in \mathcal{B}^{\geq i}_0(R)} \bigcup_{e_1 \in e(o_1)} \{(o_1,e_1):p; p \in \mathcal{L}_{e_1,i}\}$

- $\mathcal{B}_{e,i} = \mathcal{B}^{<i}_0(R) \cup \bigcup_{o_1 \in \mathcal{B}^{\geq i}_0(R)} \bigcup_{e_1 \in e(o_1)} \{(o_1,e_1):o' ; o' \in \mathcal{B}_{e_1,i}\}$

- $B_{e,i}(o) = \left\{ \begin{array}{ll} B_R(o) & \text{if } o \in \mathcal{B}^{<i}_0(R); \\ B_{e_1,i}(o') & \text{if } o = (o_1,e_1):o' \text{ with } o_1 \in \mathcal{B}^{\geq i}_0(R); \end{array} \right.$

- $b_{e,i}$ *is the extension of* $b_R|_{\mathcal{B}^{<i}_0(R)}$ *that associates with each* $(o_1,e_1):o' \in \mathcal{B}_{e,i}$*, where* $o_1 \in \mathcal{B}^{\geq i}_0(R)$*, the function* $p \mapsto \left\{ \begin{array}{ll} (o_1,e_1):b_{e_1,i}(o')(p) & \text{if } b_{e_1,i}(o')(p) \notin \mathcal{P}^f_{\circ}(B_R(o_1)); \\ b_{e_1,i}(o')(p) & \text{if } b_{e_1,i}(o')(p) \in \mathcal{P}^f_{\circ}(B_R(o_1)). \end{array} \right.$

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[16]Boxes *of* depth $\geq i$ are boxes whose content is a proof-net of depth $\geq i$; the reader should not confuse *boxes of depth* $\geq i$ with *boxes at depth* $\geq i$.
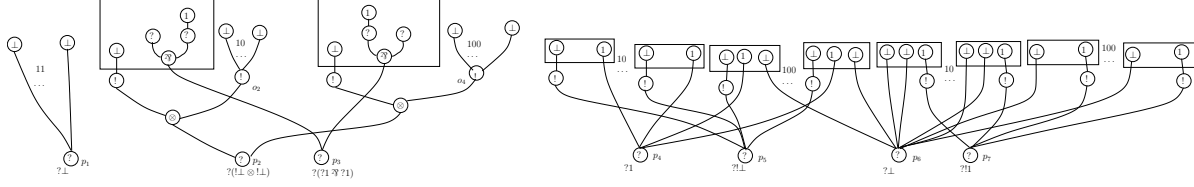
8

Figure 6: $\mathcal{T}(f)[1]$

Example 6. *If f is a pseudo-experiment of the proof-net R of Figure 1 with $f^{\#}$ like in Example 5, then Figures 5 and 6 represent respectively $\mathcal{T}(f)[0]$ and $\mathcal{T}(f)[1]$.*

The injectivity of the relational semantics for differential PS's of depth 0 is trivial (one can proceed by induction on the cardinality of the set of ports). Since $[\![\mathcal{T}(\overline{e})[0]]\!] = \{e_{|\mathcal{P}^f(R)}\}$, one can easily identify the result $e_{|\mathcal{P}^f(R)}$ of an experiment e with the differential net $\mathcal{T}(\overline{e})[0]$:

Lemma 1. *Let R and R' be two $\circ$-PS's such that $\mathcal{P}^f(R) = \mathcal{P}^f(R')$. Let e be an experiment of R and let e' be an experiment of R' such that $e_{|\mathcal{P}^f(R)} = e'_{|\mathcal{P}^f(R')}$. Then $\mathcal{T}(\overline{e})[0] \equiv \mathcal{T}(\overline{e'})[0]$.*

Now, the following fact shows that if we are able to recover $\mathcal{T}(\overline{e})[depth(R)]$ from $\mathcal{T}(\overline{e})[0]$, then we are done.

Fact 2. *Let R be a $\circ$-PS. Let e be a pseudo-experiment of R. Then $\mathcal{T}(e)[depth(R)] = R$.*

If e is a k-injective experiment of R, then, for any $i \in \mathbb{N}$, there exists a bijection $!_{e,i} : \bigcup_{o \in \mathcal{B}^{\geq i}(R)} \{\log_k(m); m \in e^{\#}(o)\} \simeq \mathcal{P}_0^!(\mathcal{T}(e)[i]) \setminus \mathcal{B}_0(\mathcal{T}(e)[i])$ such that, for any $j \in \mathsf{dom}(!_{e,i})$, we have $(a_{\mathcal{T}(e)[i]} \circ !_{e,i})(j) = k^j$. In Subsection 3.1, we will show how to recover $\bigcup_{o \in \mathcal{B}^{\geq i}(R)} \{\log_k(m); m \in e^{\#}(o)\}$ from $\mathcal{T}(e)[0]$. There are two kinds of boxes of $\mathcal{T}(e)[i+1]$ at depth 0: the "new" boxes of depth i and the boxes of depth $< i$, which are the "old" boxes (i.e. that already were in $\mathcal{T}(e)[i]$) that do not go inside some "new" box:

Fact 3. *Let R be a $\circ$-PS. Let e be a pseudo-experiment of R. Let $i \in \mathbb{N}$. Then we have*

- $\mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1]) = \mathcal{B}_0(\mathcal{T}(e)[i]) \cap \mathcal{P}_0^!(\mathcal{T}(e)[i+1])$;

- $B_{\mathcal{T}(e)[i+1]}\big|_{\mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1])} = B_{\mathcal{T}(e)[i]}\big|_{\mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1])}$;

- *and* $b_{\mathcal{T}(e)[i+1]}\big|_{\mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1])} = b_{\mathcal{T}(e)[i]}\big|_{\mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1])}$.

The challenge is the rebuilding of the "new" boxes at depth 0 of depth i.

3 From $\mathcal{T}(e)[i]$ to $\mathcal{T}(e)[i+1]$

3.1 The outline of the boxes

In this subsection we first show how to recover the set $\bigcup_{o \in \mathcal{B}^{\geq i}(R)} \{\log_k(m); m \in e^{\#}(o)\}$ and, therefore, the set $\mathcal{P}_0^!(\mathcal{T}(e)[i]) \setminus \mathcal{B}_0(\mathcal{T}(e)[i])$ (Lemma 4). Next, we show how to determine, from $\mathcal{T}(e)[i]$, the set $\mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1])$ of "new" boxes and, for any such "new" box $o \in \mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1])$, the set $\mathsf{im}(b_{\mathcal{T}(e)[i+1]}(o))$ of exponential ports that are immediately below (Proposition 5). In particular, we have $\mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1]) = \mathsf{im}(!_{e,i}|_{\mathcal{N}_i(e)})$, where the set $\mathcal{N}_i(e) \subseteq \mathbb{N}$ is defined from the set $\mathcal{M}_0(e)$ of the numbers of copies of boxes taken by the pseudo–experiment e:

Definition 10. *Let R be a differential $\circ$-PS. Let $k > 1$. Let e be a k-injective pseudo-experiment of R. For any $i \in \mathbb{N}$, we define, by induction on i, $\mathcal{M}_i(e) \subseteq \mathbb{N} \setminus \{0\}$ and $(m_{i,j}(e))_{j \in \mathbb{N}} \in \{0, \dots, k-1\}^{\mathbb{N}}$ as follows. We set $\mathcal{M}_0(e) = \bigcup_{o \in \mathcal{B}(R)} \{j \in \mathbb{N}; k^j \in e^{\#}(o)\}$ and we write $\mathsf{Card}(\mathcal{M}_i(e))$ in base k: $\mathsf{Card}(\mathcal{M}_i(e)) = \sum_{j \in \mathbb{N}} m_{i,j}(e) \cdot k^j$; we set $\mathcal{M}_{i+1}(e) = \{j > 0; m_{i,j}(e) \neq 0\}$.*

For any $i \in \mathbb{N}$, we set $\mathcal{N}_i(e) = \mathcal{M}_i(e) \setminus \mathcal{M}_{i+1}(e)$.
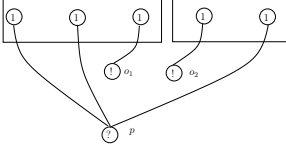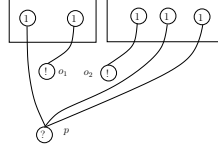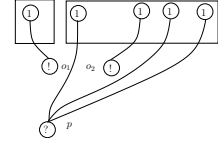
9

Figure 7: R_1

Figure 8: R_2

Figure 9: R_3

Notice that all the sets $\mathcal{M}_i(e)$ and $\mathcal{N}_i(e)$ can be computed from $\mathcal{T}(e)[0]$, since we have $\mathcal{M}_0(e) = \{a_{\mathcal{T}(e)[0]}(p); \ p \in \mathcal{P}_0^!(\mathcal{T}(e)[0])\}$.

Example 7. *If f is a 10-injective pseudo-experiment as in Example 5, then $\mathcal{M}_0(f) = \{1, \ldots, 224\}$. We have $\mathsf{Card}(\mathcal{M}_0(f)) = 4 + 2 \cdot 10^1 + 2 \cdot 10^2$, hence $\mathcal{M}_1(f) = \{1, 2\}$ and $\mathcal{N}_0(f) = \{3, \ldots, 224\}$. We have $\mathsf{Card}(\mathcal{M}_1(f)) = 2$, hence $\mathcal{M}_2(f) = \emptyset$ and $\mathcal{N}_1(f) = \{1, 2\}$.*

The following lemma shows that, for any k-injective pseudo-experiment e of R, for any $i \in \mathbb{N}$, the function $!_{e,i}$ is actually a bijection $\mathcal{M}_i(e) \to \mathcal{P}_0^!(\mathcal{T}(e)[i]) \setminus \mathcal{B}_0(\mathcal{T}(e)[i])$ such that, for any $j \in \mathcal{M}_i(e)$, we have $(a_{\mathcal{T}(e)[i]} \circ !_{\mathcal{T}(e)[i]})(j) = k^j$.

Lemma 4. *Let R be a $\circ$-PS. Let $k > \mathsf{Card}(\mathcal{B}(R))$. For any k-injective pseudo-experiment e of R, for any $i \in \mathbb{N}$, we have $\mathcal{M}_i(e) = \bigcup_{o \in \mathcal{B}^{\geq i}(R)} \{j \in \mathbb{N}; k^j \in e^{\#}(o)\}$, hence $\mathcal{N}_i(e) = \bigcup_{o \in \mathcal{B}^{=i}(R)} \{j \in \mathbb{N}; k^j \in e^{\#}(o)\}$.*

Example 8. *(Continuation of Example 7) We thus have $\mathcal{M}_1(f) = \{1, 2\}$ and $\mathcal{P}_0^!(\mathcal{T}(f)[i]) \setminus \mathcal{B}_0(\mathcal{T}(f)[1]) = \{o_2, o_4\}$ with $a_{\mathcal{T}(f)[1]}(o_2) = 10^1$ and $a_{\mathcal{T}(f)[1]}(o_4) = 10^2$ (see Figure 6).*

The set $\mathcal{K}_{k, \mathcal{N}_i(e)}(S)$ of "critical ports" is a set of exponential ports that will play a crucial role in our algorithm.

Definition 11. *Let S be a differential $\circ$-PS. Let $k > 1$. For any $p \in \mathcal{P}_0(S)$, we define the sequence $(m_{k,j}(S)(p))_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ as follows: $a_S(p) = \sum_{j \in \mathbb{N}} m_{k,j}(S)(p) \cdot k^j$. For any $j \in \mathbb{N}$, we set $\mathcal{K}_{k,j}(S) = \{p \in \mathcal{P}_0(S); m_{k,j}(S)(p) \neq 0\} \cap \mathcal{P}^e(\mathcal{G}(S))$ and, for any $J \subseteq \mathbb{N}$, we set $\mathcal{K}_{k,J}(S) = \bigcup_{j \in J} \mathcal{K}_{k,j}(S)$.*

Example 9. *We have $\mathcal{K}_{10,1}(S) = \{p_1, p_4, p_5, p_6, p_7, o_2\}$ and $\mathcal{K}_{10,2}(S) = \{p_4, p_5, p_6, p_7, o_4\}$, where S is the PS of Figure 6. So we have $\mathcal{K}_{10,\{1,2\}}(S) = \{p_1, p_4, p_5, p_6, p_7, o_2, o_4\}$.*

Critical ports are defined by their arities. We show that they are exponential ports that are immediately below the "new" boxes:

Proposition 5. *Let R be a $\circ$-PS. Let $k > \mathsf{Card}(\mathcal{B}(R)), \mathsf{cosize}(R)$. Let e be a k-injective pseudo-experiment of R and let $i \in \mathbb{N}$. Then we have $\mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1]) = \mathsf{im}(!_{e,i}|_{\mathcal{N}_{i,e}})$. Furthermore, for any $j \in \mathcal{N}_i(e)$, we have $\mathsf{im}(b_{\mathcal{T}(e)[i+1]}(!_{e,i}(j))) = \mathcal{K}_{k,j}(\mathcal{T}(e)[i])$ and, if $!_{e,i}(j) \notin \mathcal{B}_0^{=i}(R)$, then there exist $o_1 \in \mathcal{B}_0^{\geq i+1}(R)$ and $e_1 \in e(o_1)$ such that $j \in \mathcal{N}_i(e_1)$. In particular, we have $\mathcal{K}_{k, \mathcal{N}_i(e)}(\mathcal{T}(e)[i]) \subseteq \mathcal{P}_0^e(\mathcal{T}(e)[i+1])$.*

Example 10. *(Continuation of Example 8) We thus have $\mathsf{im}(!_{f,1}|_{\mathcal{N}_1(f)}) = \{o_2, o_4\}$; and indeed o_2 and o_4 are the boxes of depth 1 at depth 0 of $\mathcal{T}(f)[2] \equiv R$ (see Figure 1). Moreover we have $\mathcal{K}_{10,1}(\mathcal{T}(f)[1]) = \{p_1, p_4, p_5, p_6, p_7, o_2\}$ and $\mathcal{K}_{10,2}(\mathcal{T}(f)[1]) = \{p_4, p_5, p_6, p_7, o_4\}$; and indeed, in Figure 1, we have $\mathsf{im}(b_{\mathcal{T}(f)[2]}(o_2)) = \{p_1, p_4, p_5, p_6, p_7, o_2\}$ and $\mathsf{im}(b_{\mathcal{T}(f)[2]}(o_4)) = \{p_4, p_5, p_6, p_7, o_4\}$.*

As the following example shows, the information we obtain is already strong, but not strong enough.

Example 11. *The PS's R_1, R_2 and R_3 of Figures 7, 8 and 9 respectively have the same LPS. But if we know that $p \in \mathsf{im}(b_R(o_1))$, then we know that $R \neq R_3$. Still we are not able to distinguish between R_1 and R_2.*
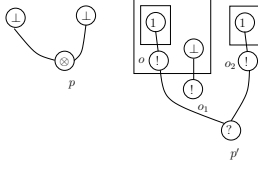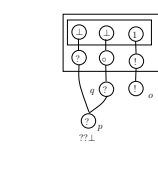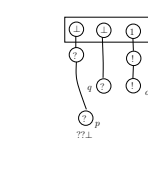
Figure 10: PS S

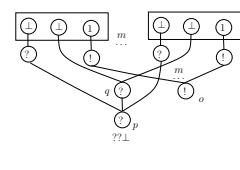Figure 11: PS R'

Figure 12: PS U Figure 13: PS $\mathcal{T}(e')[1]$ T

Figure 14: PS

3.2 Connected components

In order to rebuild the content of the boxes, we introduce our notion of *connected component* (Definition 15), which uses the auxiliary notions of *substructure* (Definition 12) and *connected substructure* (Definition 14). A differential ∘-PS R is a substructure of a differential ∘-PS S (we write $R \sqsubseteq S$) if R is obtained from S by erasing some ports and wires. More precisely:

Definition 12. *Let R and S be two differential ∘-PS's. Let $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$. We write $R \sqsubseteq_{\mathcal{Q}} S$ to denote that $\mathcal{P}_0(R) \subseteq \mathcal{P}_0(S)$, $\mathcal{W}_0(R) = \{w \in \mathcal{W}_0(S) \cap (\mathcal{P}_0(R) \setminus \mathcal{Q}); t_{\mathcal{G}(S)}(w) \in \mathcal{P}_0(R)\}$, $l_{\mathcal{G}(R)} = l_{\mathcal{G}(S)}|_{\mathcal{P}_0(R)}$, $t_{\mathcal{G}(R)} = t_{\mathcal{G}(S)}|_{\mathcal{W}_0(R)}$, $\mathcal{L}(\mathcal{G}(R)) = \mathcal{L}(\mathcal{G}(S)) \cap \{w \in \mathcal{W}_0(S); t_{\mathcal{G}(S)}(w) \in \mathcal{P}_0^m(R)\}$, $T_{\mathcal{G}(R)} = T_{\mathcal{G}(S)}|_{\mathcal{P}_0(R)}$, $\mathcal{B}_0(R) = \mathcal{B}_0(S) \cap \mathcal{P}_0(R)$, $B_R = B_S|_{\mathcal{B}_0(R)}$ and $b_R = b_S|_{\mathcal{B}_0(R)}$. We write $R \sqsubseteq S$ if there exists $\mathcal{Q}$ such that $R \sqsubseteq_{\mathcal{Q}} S$.*

Remark 1. *Let $R \sqsubseteq S$ and $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$. We have $R \sqsubseteq_{\mathcal{Q}} S$ iff $\mathcal{Q} \cap \mathcal{P}_0(R) \subseteq \mathcal{P}^f(R)$.*

Remark 2. *If $R, R' \sqsubseteq_{\mathcal{Q}} S$ and $\mathcal{P}_0(R) = \mathcal{P}_0(R')$, then $R = R'$.*

Let us explain with the following example why we sometimes need to erase some wires (so the notion of $\sqsubseteq_\emptyset$ is not enough).

Example 12. *We need to erase some wires whenever there exist a box o and $p, q \in \text{im}(b_{R'}(o))$ such that $t_{\mathcal{G}(R')}(q) = p$. Consider, for instance, Figure 11. If e' is a k-injective pseudo-experiment of R', then we want to be able to consider the PS U of Figure 12 as a substructure of $\mathcal{T}(e')[1]$, so we need to erase the wire q; we thus have $U \sqsubseteq_{\{p,q,o\}} \mathcal{T}(e')[1]$.*

The relation $\circlearrowright_S$ formalizes the notion of "connectness" between two ports of S at depth 0. But be aware that, here, "connected" has nothing to do with "connected" in the sense of [9]: here, any two doors of the same box are always "connected".

Definition 13. *Let S be a differential ∘-PS. We define the binary relation $\circlearrowright_S$ on $\mathcal{P}_0(S)$ as follows: for any $p, p' \in \mathcal{P}_0(S)$, we have $p \circlearrowright_S p'$ iff $(p \in \mathcal{W}_0(S)$ and $p' = t_{\mathcal{G}(S)}(p))$ or $(p' \in \mathcal{W}_0(S)$ and $p = t_{\mathcal{G}(S)}(p'))$ or $((\exists o \in \mathcal{B}_0(S))\{p, p'\} \subseteq \text{im}(b_S(o)))$.*

Definition 14. *Let S and T be two differential ∘-PS's. Let $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$ such that $T \sqsubseteq_{\mathcal{Q}} S$. We write $T \trianglelefteq_{\mathcal{Q}} S$ if, for any $p, p' \in \mathcal{P}_0(T)$, there exists a finite sequence $(p_0, \ldots, p_n)$ of elements of $\mathcal{P}_0(T)$ such that $p_0 = p$, $p_n = p'$ and, for any $j \in \{0, \ldots, n-1\}$, we have $p_j \circlearrowright_S p_{j+1}$ and $(p_j \in \mathcal{Q} \Rightarrow j = 0)$.*

Remark 3. *If $T \trianglelefteq_{\mathcal{Q}} S$, $\mathcal{Q}' \subseteq \mathcal{P}_0^e(S)$ and $\mathcal{P}_0(T) \cap \mathcal{Q}' \subseteq \mathcal{Q}$, then $T \trianglelefteq_{\mathcal{Q}'} S$.*

The sets $\mathcal{S}_S^k((\mathcal{Q}, \mathcal{Q}_0))$ of "components T of S above $\mathcal{Q}$ and $\mathcal{Q}_0$ that are connected *via* other ports than $\mathcal{Q}$ and such that $cosize(T) < k$" will play a crucial role in the algorithm of the rebuilding of $\mathcal{T}(\overline{e})[i+1]$ from $\mathcal{T}(\overline{e})[i]$. The reader already knows that, here, "connected" has nothing to do with the "connected proof-nets" of [9]: there, the crucial tool used was rather the "bridges" that put together two doors of the same copy of some box only if they are connected in the LPS of the proof-net.

Definition 15. *Let $k \in \mathbb{N}$. Let S be a differential ∘-PS. Let $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$ and $\mathcal{Q}_0 \subseteq \mathcal{P}_0(S)$. We set*

$$\mathcal{S}_S^k((\mathcal{Q}, \mathcal{Q}_0)) = \left\{ T \trianglelefteq_{\mathcal{Q}} S; \begin{array}{c} cosize(T) < k \text{ and } \mathcal{P}^f(T) \subseteq \mathcal{Q} \cup \mathcal{Q}_0 \text{ and} \\ \mathcal{P}_0(T) \setminus \mathcal{Q} \neq \emptyset \text{ and } (\forall p \in \mathcal{P}_0(T))(\forall q \in \mathcal{P}_0(S)) \\ ((p \circlearrowright_S q \text{ and } q \notin \mathcal{P}_0(T)) \Rightarrow p \in \mathcal{Q}) \end{array} \right\}.$$

We write also $\mathcal{S}_S^k(\mathcal{Q})$ instead of $\mathcal{S}_S^k((\mathcal{Q}, \emptyset))$ and $\mathcal{C}^k(S) = \mathcal{S}_S^k((\mathcal{P}_\circ^f(S), \mathcal{P}_\bullet^f(S)))$.

11

A port at depth 0 of S that is not in $\mathcal{Q}$ cannot belong to two different components:

Fact 6. *Let $k \in \mathbb{N}$. Let S be a differential $\circ$-PS. Let $\mathcal{Q}, \mathcal{Q}_0 \subseteq \mathcal{P}_0(S)$. Let $T, T' \in \mathcal{S}_S^k((\mathcal{Q}, \mathcal{Q}_0))$ such that $(\mathcal{P}_0(T) \cap \mathcal{P}_0(T')) \setminus \mathcal{Q} \neq \emptyset$. Then $T = T'$.*

Example 13. *We have $\mathsf{Card}\left(\mathcal{S}_S^{10}(\{p_1, p_4, p_5, p_6, p_7, o_2\})\right) = 241$ and $\mathsf{Card}\left(\mathcal{S}_S^{10}(\{p_4, p_5, p_6, p_7, o_4\})\right) = 320$, where S is the PS of Figure 6.*

The operator $\sum$ glues together several $\circ$-PS's that share only $\circ$-conclusions:

Definition 16. *Let $\mathcal{U}$ be a set of $\circ$-PS's. We say that $\mathcal{U}$ is gluable if, for any $R, S \in \mathcal{U}$ s.t. $R \neq S$, we have $\mathcal{P}_0(R) \cap \mathcal{P}_0(S) \subseteq \mathcal{P}_\circ^f(R) \cap \mathcal{P}_\circ^f(S)$. If $\mathcal{U}$ is gluable, then $\sum \mathcal{U}$ is the $\circ$-PS such that $\mathcal{P}_\circ^f(\sum \mathcal{U}) = \bigcup\{\mathcal{P}_\circ^f(R); R \in \mathcal{U}\}$ obtained by glueing all the elements of $\mathcal{U}$.*

The set $\mathcal{C}^k(R)$ (for k big enough) is an alternative way to describe a $\circ$-PS R:

Fact 7. *Let R be a $\circ$-PS. Let $k > cosize(R)$. We have $R = \sum \mathcal{C}^k(R)$.*

Definition 17 allows to formalize the operation of "putting a connected component inside a box", which will be useful for building the boxes of depth i of $\mathcal{T}(\overline{e})[i+1]$: from some *boxable* differential $\circ$-PS $R \sqsubseteq \mathcal{T}(\overline{e})[i]$, we build a $\circ$-PS $\overline{R}$ such that, for some $o \in \mathcal{B}_0^{=i}(\mathcal{T}(\overline{e})[i+1])$, there exists $T \in \mathcal{C}^k(B_{\mathcal{T}(\overline{e})[i+1]}(o))$ such that $T \simeq \overline{R}$.

Definition 17. *Let R be a differential $\circ$-PS. If $\mathcal{P}^f(R) \subseteq \mathcal{P}_0^e(R)$, $\mathcal{P}^f(R) \cap \mathcal{B}_0(R) = \emptyset$ and $\mathcal{P}_0^!(R) \setminus \mathcal{B}_0(R) \subseteq \mathcal{P}^f(R)$, then one says that R is boxable and we define a $\circ$-PS $\overline{R}$ s.t. $\mathcal{P}_0(\overline{R}) \subseteq \mathcal{P}_0(R)$, $\mathcal{P}_\bullet^f(\overline{R}) \subseteq \mathcal{W}_0(R)$ and $\mathcal{P}_\circ^f(\overline{R}) = \mathcal{P}^f(R) \cap \mathcal{P}_0(\overline{R})$ as follows:*

- *$\mathcal{P}_0(\overline{R}) = \mathcal{W}_0(R) \cup \bigcup_{o \in \mathcal{B}_0(R)} (im(b_R(o)) \cap \mathcal{P}^f(R))$;*

- *$\mathcal{W}_0(\overline{R}) = \{w \in \mathcal{W}_0(R); t_{\mathcal{G}(R)}(w) \in \mathcal{W}_0(R)\}$*

- *$l_{\mathcal{G}(\overline{R})}(p) = \begin{cases} l_{\mathcal{G}(R)}(p) & \text{if } p \in \mathcal{W}_0(R); \\ \circ & \text{otherwise}; \end{cases}$*

- *$t_{\mathcal{G}(\overline{R})} = t_{\mathcal{G}(R)}|_{\mathcal{W}_0(\overline{R})}$; $\mathcal{L}(\mathcal{G}(\overline{R})) = \mathcal{L}(\mathcal{G}(R)) \cap \mathcal{W}_0(\overline{R})$; $\mathcal{B}_0(\overline{R}) = \mathcal{B}_0(R)$; $b_{\overline{R}} = b_R$.*

If $\mathcal{U}$ is a set of boxable differential $\circ$-PS's, then we set $\overline{\mathcal{U}} = \{\overline{R}; R \in \mathcal{U}\}$.

In the proof of the following proposition, we finally describe the complete algorithm leading from $\mathcal{T}(e)[i]$ to $\mathcal{T}(e)[i+1]$. Informally: for every $j_0 \in \mathcal{N}_i(e)$, for every equivalence class $\mathfrak{T} \in \mathcal{S}_{\mathcal{T}(e)[i]}^k(\mathcal{K}_{k,j_0}(\mathcal{T}(e)[i]))_{/\equiv}$, if $\mathsf{Card}(\mathfrak{T}) = \sum_{j \in \mathbb{N}} m_j \cdot k^j$ (with $0 \leq m_j < k$), then we remove $m_{j_0} \cdot k^{j_0}$ elements of $\mathfrak{T}$ from $\mathcal{G}(\mathcal{T}(e)[i])$ and we put m_{j_0} such elements inside the (new) box $!_{e,i}(j_0)$ of depth i. For every $j_0 \in \mathcal{N}_i(e)$, the set $\mathcal{U}_{j_0}$ of the proof is the union of the sets of such m_{j_0} elements for all the equivalence classes $\mathfrak{T} \in \mathcal{S}_{\mathcal{T}(e)[i]}^k(\mathcal{K}_{k,j_0}(\mathcal{T}(e)[i]))_{/\equiv}$.

Proposition 8. *Let R and R' be two PS's. Let $k > cosize(R), cosize(R'), \mathsf{Card}(\mathcal{B}(R)), \mathsf{Card}(\mathcal{B}(R'))$. Let e be a k-injective pseudo-experiment of R and let e' be a k-injective pseudo-experiment of R' s.t. $\mathcal{T}(e)[0] \equiv \mathcal{T}(e')[0]$. Then, for any $i \in \mathbb{N}$, we have $\mathcal{T}(e)[i] \equiv \mathcal{T}(e')[i]$.*

Proof. (Sketch) By induction on i. We assume that $\mathcal{T}(e)[i] \equiv \mathcal{T}(e')[i]$. We set $\mathcal{M} = \mathcal{M}_i(e) = \mathcal{M}_i(e')$ and $\mathcal{N} = \mathcal{N}_i(e) = \mathcal{N}_i(e')$.

Let $S \equiv \mathcal{T}(e)[i]$. There is a bijection $! : \mathcal{M} \to \mathcal{P}_0^!(S) \setminus \mathcal{B}_0(S)$ such that, for any $j \in \mathcal{M}$, we have $(a_S \circ !)(j) = k^j$. For any $j \in \mathcal{N}$, we set $\mathcal{K}_j = \mathcal{K}_{k,j}(S)$ and $\mathcal{T}_j = \mathcal{S}_S^k(\mathcal{K}_j)$. We set $\mathcal{T} = \bigcup_{j \in \mathbb{N}} \mathcal{T}_j$. For any $T \in \mathcal{T}$, we define $(m_j^T)_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ as follows: $\mathsf{Card}(\{T' \in \mathcal{T}; T \equiv T'\}) = \sum_{j \in \mathbb{N}} m_j^T \cdot k^j$. We set $\mathcal{P} = \{p \in \mathcal{P}_0(S); p \notin \bigcup_{j \in \mathcal{N}} \bigcup_{T \in \mathcal{S}_S^k(\mathcal{K}_j)} \mathcal{P}_0(T)\}$. For any $j \in \mathcal{N}$, we are given $\mathcal{U}_j \subseteq \mathcal{T}_j$ such that, for any $T \in \mathcal{T}_j$, we have $\mathsf{Card}(\{T' \in \mathcal{U}_j; T' \equiv T\}) = m_j^T$. Let S' be some differential PS such that $\mathcal{G}(S|_{\mathcal{P}}) \sqsubseteq_\emptyset \mathcal{G}(S') \sqsubseteq_\emptyset \mathcal{G}(S)$, where $S|_{\mathcal{P}}$ is the unique $S_0 \sqsubseteq_\emptyset S'$ s.t. $\mathcal{P}_0(S_0) = \mathcal{P}$, and:

- $\bigcup_{j_0 \in \mathcal{N}} \mathcal{T}'_{j_0} \subseteq \bigcup_{j_0 \in \mathcal{N}} \mathcal{T}_{j_0}$

- for any $T \in \mathcal{T}$, we have $\mathsf{Card}\left(\{T' \in \bigcup_{j \in \mathbb{N}} \mathcal{S}^k_{S'}(\mathcal{K}_j); T' \equiv T\}\right) = \sum_{j \notin \mathcal{N}} m^T_j \cdot k^j$;

- $\mathcal{B}_0(S') = (\mathcal{B}_0(S) \cap \mathcal{P}_0(S')) \cup \mathsf{im}(!|_{\mathcal{N}})$

- for any $o_1 \in \mathcal{B}_0(S) \cap \mathcal{P}_0(S')$, we have $B_{S'}(o_1) = B_S(o_1)$ and $b_{S'}(o_1) = b_S(o_1)$

- for any $j \in \mathcal{N}$, there exists $\rho^j : B_{S'}(!(j)) \simeq \sum \overline{\mathcal{U}_j}$ such that $b_{S'}(!(j))(q) = $
$\begin{cases} \mathcal{G}(\rho^j)(q) & \text{if } q \in \mathcal{P}^f_\circ(B_{S'}(!(j))); \\ t_{\mathcal{G}(S)}(\mathcal{G}(\rho^j)(q)) & \text{if } q \in \mathcal{P}^f_\bullet(B_{S'}(!(j))). \end{cases}$

Then one can show that $\mathcal{T}(e)[i+1] \equiv S' \equiv \mathcal{T}(e')[i+1]$. $\qquad\square$

Example 14. *Consider Figure 6. We set $j_0 = 1$. We have $o_2 = !_{f,1}(j_0)$. We set $\mathcal{T}_{j_0} = \mathcal{S}^{10}_{\mathcal{T}(f)[1]}(\mathcal{K}_{10,1}(\mathcal{T}(f)[1]))$. Let T be the $\circ$-PS of Figure 14. We have $T \in \mathcal{T}$. We have $\mathsf{Card}(\{T' \in \mathcal{T}; T' \equiv T\}) = 1 \cdot 10^0 + 1 \cdot 10^1$. We set $\mathcal{U}_{j_0} = \{T\}$. The $\circ$-PS $\sum \overline{\mathcal{U}_{j_0}} = \overline{T}$ is the $\circ$-PS of depth 0 that consists of only one port: the port p labelled by $\perp$. There exists $\rho^{j_0} : B_{\mathcal{T}(f)[2]}(o_2) \simeq \sum \overline{\mathcal{U}_{j_0}}$ such that $b_{\mathcal{T}(f)[2]}(o_2)(q_2) = p_1 = t_{\mathcal{G}(\mathcal{T}(f)[1])}(q)$ defined by $\mathcal{G}(\rho^{j_0})(q_2) = q$ (see Figure 1).*

3.3 Injectivity

Theorem 9. *Let R and R' be two PS's s.t. $\mathcal{P}^f(R) = \mathcal{P}^f(R')$. If $[\![R]\!] = [\![R']\!]$, then $R \equiv R'$.*

Proof. We set $d = \max\{depth(R), depth(R')\}$. For any $k > 1$, there exist a k-injective experiment e of R and a k-injective experiment e' of R'[17] such that $e_{|\mathcal{P}^f(R)} = e'_{|\mathcal{P}^f(R')} \in [\![R]\!] \cap [\![R']\!]$. By Lemma 1, we have $\mathcal{T}(\overline{e})[0] \equiv \mathcal{T}(\overline{e'})[0]$. Therefore if $k > cosize(R), \mathsf{Card}(\mathcal{B}(R))$, then, by Proposition 8, we have $\mathcal{T}(\overline{e})[d] \equiv \mathcal{T}(\overline{e'})[d]$. Now, by Fact 2, we have $R = \mathcal{T}(\overline{e})[d] \equiv \mathcal{T}(\overline{e'})[d] = R'$. $\qquad\square$

Remark 4. *From any 1-point of $[\![R]\!]$ (i.e. the result of some 1-experiment of R) one can recover $cosize(R)$ and $\mathsf{Card}(\mathcal{B}(R))$. This remark shows that for characterizing R, two points are enough: a 1-point of its interpretation, from which one can bound $cosize(R)$ and $\mathsf{Card}(\mathcal{B}(R))$, and a k-injective point of its interpretation with $k > cosize(R), \mathsf{Card}(\mathcal{B}(R))$.*

Remark 5. *If we want to extend our theorem to PS's with axioms, then we assume that the interpretation of any ground type is an infinite set and we consider a k-injective experiment e that is "injective" in the sense that every atom (the atoms are the elements of the interpretations of the ground types) occurring in $\mathsf{im}(e_{|\mathcal{P}^f(R)})$ occurs exactly twice.*

Remark 6. *In an untyped framework with axioms, we need to add the constraint on the injective k-injective point one considers to be $[\![R]\!]$-atomic, i.e. a point of $[\![R]\!]$ that cannot be obtained from another point of $[\![R]\!]$ by some substitution that is not a renaming. Atomic points are results of atomic experiments (experiments that label axioms with atoms) - the converse does not necessarily hold.*

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A   Proof of Lemma 4

Fact 10. *Let R be a $\circ$-PS. Let $k > 1$. Let e be a k-injective pseudo-experiment of R. Then*

- *for any $o, o' \in \mathcal{B}_0(R)$, for any $e' \in e(o')$, we have $(\forall j \in \mathcal{M}_0(e'))e^{\#}(o) \neq \{k^j\}$;*

- *and, for any $o_1, o_2 \in \mathcal{B}_0(R)$, for any $e_1 \in e(o_1)$, for any $e_2 \in e(o_2)$, we have $\mathcal{M}_0(e_1) \cap \mathcal{M}_0(e_2) \neq \emptyset \Rightarrow (o_1 = o_2 \text{ and } e_1 = e_2)$.*

Proof. Let $o, o' \in \mathcal{B}_0(R)$ and let $e' \in e(o')$. For any $j \in \mathcal{M}_0(e')$, there exists $o'' \in \mathcal{B}(B_R(o'))$ such that $k^j \in e'^{\#}(o'')$. So, if there exists $j \in \mathcal{M}_0(e')$ such that $e^{\#}(o) = \{k^j\}$, then there exists $o'' \in \mathcal{B}(R)(o')$ such that $e^{\#}(o' : o'') \cap e^{\#}(o) \neq \emptyset$; but, by the definition of *k-injective experiment* (Definition 8), this entails that $o = o' : o''$; now, since $\varepsilon \notin \mathcal{B}(B_R(o''))$, we obtain a contradiction with the following requirement of the definition of $\circ$-PS's (Definition 2): no $p \in \mathcal{P}(\mathcal{G}(R))$ is a sequence. $\qquad \square$

Lemma 11. *Let $k > \mathsf{Card}\,(\mathcal{B}(R))$. For any k-injective pseudo-experiment e of R, we have*

- *if $depth(R) \leq 1$, then $\mathcal{M}_1(e) = \emptyset$*

- *$m_{0,0}(e) = \mathsf{Card}\,(\mathcal{B}_0(R))$*

- *$\mathcal{M}_1(e) = \bigcup_{o \in \mathcal{B}_0^{\geq 1}(R)} \{j \in \mathbb{N}; e^{\#}(o) = \{k^j\}\} \cup \bigcup_{o \in \mathcal{B}_0^{\geq 2}(R)} \bigcup_{e' \in e(o)} \mathcal{M}_1(e')$*

- *$\mathcal{M}_1(e) \subseteq \mathcal{M}_0(e)$.*

Proof. The first item follows from the fact that if $depth(R) = 0$, then $\mathsf{Card}\,(\mathcal{M}_0(e)) = 0$ and if $depth(R) = 1$, then $\mathsf{Card}\,(\mathcal{M}_0(e)) = \mathsf{Card}\,(\mathcal{B}(R)) < k$.

We prove the three other items by induction on $depth(R)$. We have:

$$
\begin{aligned}
&\mathsf{Card}\,(\mathcal{M}_0(e)) \\[2mm]
=\;& \mathsf{Card}\left(\bigcup_{o \in \mathcal{B}_0(R)} \{j \in \mathbb{N}; k^j \in e^{\#}(o)\} \right) \\[2mm]
=\;& \sum_{o \in \mathcal{B}(R)} \mathsf{Card}\left(\{j \in \mathbb{N}; k^j \in e^{\#}(o)\} \right) \\[2mm]
=\;& \sum_{o \in \mathcal{B}_0(R)} \mathsf{Card}\left(\{j \in \mathbb{N}; k^j \in e^{\#}(o)\} \right) + \sum_{o \in \mathcal{B}_0^{\geq 1}(R)} \sum_{o' \in \mathcal{B}(B_R(o))} \mathsf{Card}\left(\{j \in \mathbb{N}; k^j \in e^{\#}(o : o')\} \right) \\[2mm]
=\;& \mathsf{Card}\,(\mathcal{B}_0(R)) + \sum_{o \in \mathcal{B}_0^{\geq 1}(R)} \sum_{o' \in \mathcal{B}(B_R(o))} \mathsf{Card}\left(\bigcup_{e' \in e(o)} \{j \in \mathbb{N}; k^j \in e'^{\#}(o')\} \right) \\[2mm]
=\;& \mathsf{Card}\,(\mathcal{B}_0(R)) + \sum_{o \in \mathcal{B}_0^{\geq 1}(R)} \sum_{o' \in \mathcal{B}(B_R(o))} \sum_{e' \in e(o)} \mathsf{Card}\left(\{j \in \mathbb{N}; k^j \in e'^{\#}(o')\} \right) \\[2mm]
=\;& \mathsf{Card}\,(\mathcal{B}_0(R)) + \sum_{o \in \mathcal{B}_0^{\geq 1}(R)} \sum_{e' \in e(o)} \sum_{o' \in \mathcal{B}(B_R(o))} \mathsf{Card}\left(\{j \in \mathbb{N}; k^j \in e'^{\#}(o')\} \right) \\[2mm]
=\;& \mathsf{Card}\,(\mathcal{B}_0(R)) + \sum_{o \in \mathcal{B}_0^{\geq 1}(R)} \sum_{e' \in e(o)} \mathsf{Card}\left(\bigcup_{o' \in \mathcal{B}(B_R(o))} \{j \in \mathbb{N}; k^j \in e'^{\#}(o')\} \right) \\[2mm]
& \text{(since } (\forall o \in \mathcal{B}_0^{\geq 1}(R))(\forall e' \in e(o))(\forall o' \in \mathcal{B}(B_R(o)))e'^{\#}(o') \subseteq e^{\#}(o : o')) \\[2mm]
=\;& \mathsf{Card}\,(\mathcal{B}_0(R)) + \sum_{o \in \mathcal{B}_0^{\geq 1}(R)} \sum_{e' \in e(o)} \mathsf{Card}\,(\mathcal{M}_0(e'))
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \mathsf{Card}\left(\mathcal{B}_0(R)\right) + \sum_{o\in\mathcal{B}_0^{\geq 1}(R)} \sum_{e'\in e(o)} \left(m_{0,0}(e') + \sum_{j\in\mathcal{M}_1(e')} m_{0,j}(e') \cdot k^j\right) \\
= \quad & \mathsf{Card}\left(\mathcal{B}_0(R)\right) + \sum_{o\in\mathcal{B}_0^{\geq 1}(R)} \sum_{e'\in e(o)} \left(\mathsf{Card}\left(\mathcal{B}_0(B_R(o))\right) + \sum_{j\in\mathcal{M}_1(e')} m_{0,j}(e') \cdot k^j\right)
\end{aligned}
$$

(by the induction hypothesis)

$$
\begin{aligned}
= \quad & \mathsf{Card}\left(\mathcal{B}_0(R)\right) + \sum_{o\in\mathcal{B}_0^{\geq 1}(R)} \left(\mathsf{Card}\left(\mathcal{B}_0(B_R(o))\right) \cdot \mathsf{Card}\left(e(o)\right) + \sum_{e'\in e(o)} \sum_{j\in\mathcal{M}_1(e')} m_{0,j}(e') \cdot k^j\right) \\
= \quad & \mathsf{Card}\left(\mathcal{B}_0(R)\right) + \sum_{o\in\mathcal{B}_0^{\geq 1}(R)} \mathsf{Card}\left(\mathcal{B}_0(B_R(o))\right) \cdot \mathsf{Card}\left(e(o)\right) + \sum_{o\in\mathcal{B}_0^{\geq 2}(R)} \sum_{e'\in e(o)} \sum_{j\in\mathcal{M}_1(e')} m_{0,j}(e') \cdot k^j
\end{aligned}
$$

(by the first item).

By the induction hypothesis, for any $o \in \mathcal{B}_0(R)$, for any $e' \in e(o)$, we have $\mathcal{M}_1(e') \subseteq \mathcal{M}_0(e')$, hence, by Fact 10,

- for any $o \in \mathcal{B}_0(R)$, for any $o' \in \mathcal{B}_0(R)$, for any $e' \in e(o')$, we have $(\forall j \in \mathcal{M}_1(e'))e^{\#}(o) \neq \{k^j\}$;

- and, for any $o_1, o_2 \in \mathcal{B}_0(R)$, for any $e_1 \in e(o_1)$, for any $e_2 \in e(o_2)$, we have $\mathcal{M}_1(e_1) \cap \mathcal{M}_1(e_2) \neq \emptyset \Rightarrow (o_1 = o_2 \text{ and } e_1 = e_2)$.

We obtain $\mathcal{M}_1(e) = \bigcup_{o\in\mathcal{B}_0^{\geq 1}(R)}\{j \in \mathbb{N}; e^{\#}(o) = \{k^j\}\} \cup \bigcup_{o\in\mathcal{B}_0^{\geq 2}(R)} \bigcup_{e'\in e(o)} \mathcal{M}_1(e')$. Since, by the induction hypothesis, for any $o \in \mathcal{B}_0^{\geq 2}(R)$, for any $e' \in e(o)$, we have $\mathcal{M}_1(e') \subseteq \mathcal{M}_0(e')$, we obtain $\mathcal{M}_1(e) \subseteq \mathcal{M}_0(e)$. $\quad\square$

Lemma 12. *Let R be a $\circ$-PS. Let $k > \mathsf{Card}\left(\mathcal{B}(R)\right)$. For any k-injective pseudo-experiment e of R, for any $i \in \mathbb{N}$, we have*

- *for any $o, o' \in \mathcal{B}_0(R)$, for any $e' \in e(o')$, we have $(\forall j \in \mathcal{M}_i(e'))e^{\#}(o) \neq \{k^j\}$;*

- *for any $o_1, o_2 \in \mathcal{B}_0(R)$, for any $e_1 \in e(o_1)$, for any $e_2 \in e(o_2)$, we have $\mathcal{M}_i(e_1) \cap \mathcal{M}_i(e_2) \neq \emptyset \Rightarrow (o_1 = o_2 \text{ and } e_1 = e_2)$;*

- *if $depth(R) \leq i + 1$, then $\mathcal{M}_{i+1}(e) = \emptyset$;*

- *$m_{i,0}(e) = \mathsf{Card}\left(\mathcal{B}_0^{\geq i}(R)\right)$;*

- *$\mathcal{M}_{i+1}(e) = \bigcup_{o\in\mathcal{B}_0^{\geq i+1}(R)}\{j \in \mathbb{N}; e^{\#}(o) = \{k^j\}\} \cup \bigcup_{o\in\mathcal{B}_0^{\geq i+2}(R)} \bigcup_{e'\in e(o)} \mathcal{M}_{i+1}(e')$;*

- *and $\mathcal{M}_{i+1}(e) \subseteq \mathcal{M}_i(e)$.*

Proof. By induction on i.

If $i = 0$, then the items hold by Fact 10 and Lemma 11. Now, we assume that they hold for some $i \in \mathbb{N}$. Since $\mathcal{M}_{i+1}(e) \subseteq \mathcal{M}_i(e)$, we have

1. for any $o, o' \in \mathcal{B}_0(R)$, for any $e' \in e(o')$, we have $(\forall j \in \mathcal{M}_{i+1}(e'))e^{\#}(o) \neq \{k^j\}$;

2. and, for any $o_1, o_2 \in \mathcal{B}_0(R)$, for any $e_1 \in e(o_1)$, for any $e_2 \in e(o_2)$, we have $\mathcal{M}_{i+1}(e_1) \cap \mathcal{M}_{i+1}(e_2) \neq \emptyset \Rightarrow (o_1 = o_2 \text{ and } e_1 = e_2)$.

If $depth(R) \leq i+2$, then, since, $\mathcal{M}_{i+1}(e) = \bigcup_{o\in\mathcal{B}_0^{\geq i+1}(R)}\{j \in \mathbb{N}; e^{\#}(o) = \{k^j\}\} \cup \bigcup_{o\in\mathcal{B}_0^{\geq i+2}(R)} \bigcup_{e'\in e(o)} \mathcal{M}_{i+1}(e')$, we obtain $\mathsf{Card}\left(\mathcal{M}_{i+1}(e)\right) = \mathsf{Card}\left(\bigcup_{o\in\mathcal{B}_0^{\geq i+1}(R)}\{j \in \mathbb{N}; e^{\#}(o) = \{k^j\}\}\right) < k$, hence $\mathcal{M}_{i+2}(e) = \emptyset$.

Now, we prove, by by induction on $depth(R)$, that:

- $m_{i+1,0}(e) = \mathsf{Card}\left(\mathcal{B}_0^{\geq i+1}(R)\right)$

17

- $\mathcal{M}_{i+2}(e) = \bigcup_{o \in \mathcal{B}_0^{\geq i+2}(R)}\{j \in \mathbb{N}; e^\#(o) = \{k^j\}\} \cup \bigcup_{o \in \mathcal{B}_0^{\geq i+3}(R)}\bigcup_{e' \in e(o)}\mathcal{M}_{i+2}(e')$

- and $\mathcal{M}_{i+2}(e) \subseteq \mathcal{M}_{i+1}(e)$.

We have

$$
\begin{aligned}
&\mathsf{Card}\left(\mathcal{M}_{i+1}(e)\right)\\
=\ &\mathsf{Card}\left(\mathcal{B}_0^{\geq i+1}(R)\right) + \sum_{o \in \mathcal{B}_0^{\geq i+2}(R)}\sum_{e' \in e(o)}\mathsf{Card}\left(\mathcal{M}_{i+1}(e')\right)\\
&\text{(by 1. and 2.)}\\
=\ &\mathsf{Card}\left(\mathcal{B}_0^{\geq i+1}(B_R(o))\right) + \sum_{o \in \mathcal{B}_0^{\geq i+2}(R)}\sum_{e' \in e(o)}\left(m_{i+1,0}(e') + \sum_{j \in \mathcal{M}_{i+2}(e')}m_{i+1,j}(e')\cdot k^j\right)\\
=\ &\mathsf{Card}\left(\mathcal{B}_0^{\geq i+1}(R)\right) + \sum_{o \in \mathcal{B}_0^{\geq i+2}(R)}\sum_{e' \in e(o)}\left(\mathsf{Card}\left(\mathcal{B}_0^{\geq i+1}(B_R(o))\right) + \sum_{j \in \mathcal{M}_{i+2}(e')}m_{i+1,j}(e')\cdot k^j\right)\\
&\text{(by the induction hypothesis)}\\
=\ &\mathsf{Card}\left(\mathcal{B}_0^{\geq i+1}(R)\right) + \sum_{o \in \mathcal{B}_0^{\geq i+2}(R)}\left(\mathsf{Card}\left(\mathcal{B}_0^{\geq i+1}(B_R(o))\right)\cdot \mathsf{Card}\left(e(o)\right) + \sum_{e' \in e(o)}\sum_{j \in \mathcal{M}_{i+2}(e')}m_{i+1,j}(e')\cdot k^j\right)\\
=\ &\mathsf{Card}\left(\mathcal{B}_0^{\geq i+1}(R)\right) + \sum_{o \in \mathcal{B}_0^{\geq i+2}(R)}\mathsf{Card}\left(\mathcal{B}_0^{\geq i+1}(B_R(o))\right)\cdot \mathsf{Card}\left(e(o)\right)\\
&\quad + \sum_{o \in \mathcal{B}_0^{\geq i+3}(R)}\sum_{e' \in e(o)}\sum_{j \in \mathcal{M}_{i+2}(e')}m_{i+1,j}(e')\cdot k^j\\
&\text{\textit{(since, if }depth(B_R(o)) = i+2 \text{ \textit{and} } e' \in e(o), \text{ \textit{then} } \mathcal{M}_{i+2}(e') = \emptyset).}
\end{aligned}
$$

By the induction hypothesis, for any $o \in \mathcal{B}_0^{\geq i+3}(R)$, for any $e' \in e(o)$, we have $\mathcal{M}_{i+2}(e') \subseteq \mathcal{M}_{i+1}(e')$, hence, by 1. and 2.,

- for any $o, o' \in \mathcal{B}_0(R)$, for any $e' \in e(o')$, we have $(\forall j \in \mathcal{M}_{i+2}(e'))e^\#(o) \neq \{k^j\}$;

- and, for any $o_1, o_2 \in \mathcal{B}_0(R)$, for any $e_1 \in e(o_1)$, for any $e_2 \in e(o_2)$, we have $\mathcal{M}_{i+2}(e_1) \cap \mathcal{M}_{i+2}(e_2) \neq \emptyset \Rightarrow (o_1 = o_2 \text{ and } e_1 = e_2)$.

We obtain

- $m_{i+1,0}(e) = \mathsf{Card}\left(\mathcal{B}_0^{\geq i+1}(R)\right)$

- and $\mathcal{M}_{i+2}(e) = \bigcup_{o \in \mathcal{B}_0^{\geq i+2}(R)}\{j \in \mathbb{N}; e^\#(o) = \{k^j\}\} \cup \bigcup_{o \in \mathcal{B}_0^{\geq i+3}(R)}\bigcup_{e' \in e(o)}\mathcal{M}_{i+2}(e')$.

Since, by the induction hypothesis, for any $o \in \mathcal{B}_0^{\geq i+3}(R)$, for any $e' \in e(o)$, we have $\mathcal{M}_{i+2}(e') \subseteq \mathcal{M}_{i+1}(e')$, we obtain $\mathcal{M}_{i+2}(e) \subseteq \mathcal{M}_{i+1}(e)$. $\square$

Proof of Lemma 4

Proof. By induction on $depth(R)$. We have

$$
\begin{aligned}
\mathcal{M}_i(e) \ =\ &\bigcup_{o \in \mathcal{B}_0^{\geq i}(R)}\{j \in \mathbb{N}; e^\#(o) = \{k^j\}\} \cup \bigcup_{o \in \mathcal{B}_0^{\geq i+1}(R)}\bigcup_{e' \in e(o)}\mathcal{M}_i(e')\\
&\text{\textit{(by Lemma 12)}}\\
=\ &\bigcup_{o \in \mathcal{B}_0^{\geq i}(R)}\{j \in \mathbb{N}; e^\#(o) = \{k^j\}\} \cup \bigcup_{o \in \mathcal{B}_0^{\geq i+1}(R)}\bigcup_{e' \in e(o)}\bigcup_{o' \in \mathcal{B}^{\geq i}(B_R(o))}\{j \in \mathbb{N}; k^j \in e'^\#(o')\}\\
&\text{\textit{(by the induction hypothesis)}}
\end{aligned}
$$

18

$$= \bigcup_{o \in \mathcal{B}_0^{\geq i}(R)} \{j \in \mathbb{N}; e^{\#}(o) = \{k^j\}\} \cup \bigcup_{o \in \mathcal{B}_0^{\geq i+1}(R)} \bigcup_{o' \in \mathcal{B}^{\geq i}(B_R(o))} \{j \in \mathbb{N}; k^j \in e^{\#}(o : o')\}$$

$$= \bigcup_{o \in \mathcal{B}^{\geq i}(R)} \{j \in \mathbb{N}; k^j \in e^{\#}(o)\}.$$

$\square$

B Proof of Proposition 5

We prove Proposition 5 through Lemma 13.

Definition 18. *Let R be a $\circ$-PS. For any $q \in \mathcal{P}_0(R)$, for any $i \in \mathbb{N}$, we set $a_{R,i}(q) = a_{\mathcal{G}(R)}(q) +$
$\sum_{\substack{o \in \mathcal{B}_0^{<i}(R) \\ q \in \mathcal{P}_o^f(B_R(o))}} a_{B_R(o)}(q)$
$+ \sum_{o \in \mathcal{B}_0^{<i}(R)} \mathsf{Card}\left(\{p \in \mathcal{P}_\bullet^f(B_R(o)); b_R(o)(p) = q\}\right).$*

For any $p \in \mathcal{P}^e(\mathcal{G}(R))$, for any $i \in \mathbb{N}$, we define, by induction on $\mathrm{depth}(R)$, a subset $\mathcal{B}_R^{\geq i}(p)$ of $\mathcal{B}^{\geq i}(R)$: we set

$$\begin{aligned}
\mathcal{B}_R^{\geq i}(p) &= \{o_1 \in \mathcal{B}_0^{\geq i}(R); p \in \mathrm{im}(b_R(o_1)|_{\mathcal{P}_\bullet^f(B_R(o_1))}) \\
&\quad \text{or } (p \in \mathcal{P}_o^f(B_R(o_1)) \text{ and } a_{B_R(o_1),i}(p) > 0)\} \\
&\quad \cup \bigcup_{\substack{o_1 \in \mathcal{B}_0^{\geq i+1}(R) \\ p \in \mathcal{P}_o^f(B_R(o_1))}} \{o_1 : o; o \in \mathcal{B}_{B_R(o_1)}^{\geq i}(p)\}
\end{aligned}$$

Remark 7. *If $o_1 \in \mathcal{B}_0^{=i}(R)$, then $\mathrm{im}(b_R(o_1)) = \{p \in \mathcal{P}^e(\mathcal{G}(R)); o_1 \in \mathcal{B}_R^{\geq i}(p)\}$.*

Lemma 13. *Let R be a $\circ$-PS. Let $k > \mathrm{cosize}(R)$. Let e be a k-injective pseudo-experiment of R. Let $q \in \mathcal{P}_0(R)$. Let $i \in \mathbb{N}$. Then, for any $j > 0$, we have $q \in \mathcal{K}_{k,j}(\mathcal{T}(e)[i])$ if, and only if, there exists $o \in \mathcal{B}_R^{\geq i}(q)$ such that $k^j \in e^{\#}(o)$. Moreover we have $a_{\mathcal{T}(e)[i]}(q) \bmod k = a_{R,i}(q)$.*

B.1 Proof of Lemma 13

Fact 14. *Let R be a $\circ$-PS. Let e be a pseudo-experiment of R. Let $p \in \mathcal{P}_0(R)$. Let $i \in \mathbb{N}$. If $p \notin \mathcal{P}_o^f(R)$, then*

$$\begin{aligned}
a_{\mathcal{T}(e)[i]}(p) &= a_{R,i}(p) \\
&\quad + \sum_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \mathsf{Card}\left(\{q \in \mathcal{P}_\bullet^f(B_R(o_1)); b_R(o_1)(q) = p\}\right) \cdot \mathsf{Card}(e(o_1))
\end{aligned}$$

If $p \in \mathcal{P}_o^f(R)$, then

$$\begin{aligned}
a_{\mathcal{T}(e)[i]}(p) &= a_{R,i}(p) \\
&\quad + \sum_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \mathsf{Card}\left(\{q \in \mathcal{P}_\bullet^f(B_R(o_1)); b_R(o_1)(q) = p\}\right) \cdot \mathsf{Card}(e(o_1)) \\
&\quad + \sum_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \sum_{e_1 \in e(o_1)} a_{\mathcal{T}(e_1)[i]}(p)
\end{aligned}$$

19

Proof. For any $o_1 \in \mathcal{B}_0^{\geq i}(R)$, for any $e_1 \in e(o_1)$, we have

$$
\{p \in \mathcal{P}_\bullet^f(B_{\mathcal{T}(e)[i]}((o_1, e_1) : o')); b_{\mathcal{T}(e)[i]}((o_1, e_1) : o')(p) = c\}
$$
$$
= \bigcup_{\substack{q \in \mathcal{P}_\circ^f(B_R(o_1)) \\ b_R(o_1)(p) = c}} \{p \in \mathcal{P}_\bullet^f(B_{\mathcal{T}(e_1)[i]}(o')); b_{\mathcal{T}(e_1)[i]}(o')(p) = q\}
$$

hence

$$
\mathsf{Card}\left(\{p \in \mathcal{P}_\bullet^f(B_{\mathcal{T}(e)[i]}((o_1, e_1) : o')); b_{\mathcal{T}(e)[i]}((o_1, e_1) : o')(p) = c\}\right)
$$
$$
= \sum_{\substack{q \in \mathcal{P}_\circ^f(B_R(o_1)) \\ b_R(o_1)(p) = c}} \mathsf{Card}\left(\{p \in \mathcal{P}_\bullet^f(B_{\mathcal{T}(e_1)[i]}(o')); b_{\mathcal{T}(e_1)[i]}(o')(p) = q\}\right)
$$

$\square$

Lemma 15. *Let R be a $\circ$-PS. Let $k > cosize(R)$. Let e be a k-injective pseudo-experiment of R. Let $p \in \mathcal{P}_0(R)$. Let $i \in \mathbb{N}$. Then $a_{\mathcal{T}(e)[i]}(p) \bmod k = a_{R,i}(p)$.*

Proof. First, notice that
$$
a_{R,i}(p) \leq a_R(p) \leq cosize(R) < k \quad (*)
$$

Now, we prove the statement by induction on $depth(R)$. If $depth(R) = 0$, then we just apply $(*)$. If $depth(R) > 0$, then, by induction hypothesis, we have

$$
(\sum_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \sum_{e_1 \in e(o_1)} \sum_{\substack{q \in \mathcal{P}_\circ^f(B_R(o_1)) \\ b_R(o_1)(q) = p}} a_{\mathcal{T}(e_1)[i]}(q)) \bmod k
$$
$$
= (\sum_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \sum_{e_1 \in e(o_1)} \sum_{\substack{q \in \mathcal{P}_\circ^f(B_R(o_1)) \\ b_R(o_1)(q) = p}} a_{B_R(o_1)}(q)) \bmod k
$$
$$
= (\sum_{o_1 \in \mathcal{B}_0^{\geq i}(R)} (\sum_{\substack{q \in \mathcal{P}_\circ^f(B_R(o_1)) \\ b_R(o_1)(q) = p}} a_{B_R(o_1)}(q)) \cdot \mathsf{Card}\,(e(o_1))) \bmod k
$$
$$
= 0
$$

hence

$$
\begin{aligned}
a_{\mathcal{T}(e)[i]}(p) \bmod k \;=\; & (a_{R,i}(p) \\
& + \sum_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \mathsf{Card}\left(\{q \in \mathcal{P}_\bullet^f(B_R(o_1)); b_R(o_1)(q) = p\}\}\right) \cdot \mathsf{Card}\,(e(o_1)) \\
& + \sum_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \sum_{e_1 \in e(o_1)} \sum_{\substack{q \in \mathcal{P}_\circ^f(B_R(o_1)) \\ b_R(o_1)(q) = p}} a_{\mathcal{T}(e_1)[i]}(q)) \bmod k \\
& \text{(by Fact 14)} \\
=\; & a_{R,i}(p)
\end{aligned}
$$

$\square$

Proof of Lemma 13:

Proof. By Lemma 15, we already know that $a_{\mathcal{T}(e)[i]}(q) \bmod k = a_{R,i}(q)$. We prove, by induction on $depth(R)$, that, for any $j > 0$, we have $m_{k,j}(\mathcal{T}(e)[i])(q) > 0$ if, and only if, there exists $o \in \mathcal{B}_R^{\geq i}(q)$ such that $k^j \in e^{\#}(o)$. If $depth(R) > 0$, then we have

$$
\begin{aligned}
& a_{\mathcal{T}(e)[i]}(q) \\
= \; & a_{\mathcal{G}(R)}(q) \\
& + \sum_{o_1 \in \mathcal{B}_0^{<i}(R)} \mathsf{Card}\left(\{p \in \mathcal{P}_{\bullet}^{\mathsf{f}}(B_R(o_1)); b_R(o_1)(p) = q\}\right) \\
& + \sum_{\substack{o_1 \in \mathcal{B}_0^{<i}(R) \\ q \in \mathcal{P}_{\circ}^{\mathsf{f}}(B_R(o_1))}} a_{B_R(o_1)}(q) \\
& + \sum_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \left(\mathsf{Card}\left(e(o_1)\right) \cdot \mathsf{Card}\left(\{p \in \mathcal{P}_{\bullet}^{\mathsf{f}}(B_R(o_1)); b_R(o_1)(p) = q\}\right)\right) \\
& + \sum_{\substack{o_1 \in \mathcal{B}_0^{=i}(R) \\ q \in \mathcal{P}_{\circ}^{\mathsf{f}}(B_R(o_1))}} \sum_{e_1 \in e(o_1)} a_{\mathcal{T}(e_1)[i]}(q) + \sum_{\substack{o_1 \in \mathcal{B}_0^{\geq i+1}(R) \\ q \in \mathcal{P}_{\circ}^{\mathsf{f}}(B_R(o_1))}} \sum_{e_1 \in e(o_1)} a_{\mathcal{T}(e_1)[i]}(q) \\
= \; & a_{\mathcal{G}(R)}(q) \\
& + \sum_{o_1 \in \mathcal{B}_0^{<i}(R)} \left(\mathsf{Card}\left(\{p \in \mathcal{P}_{\bullet}^{\mathsf{f}}(B_R(o_1)); b_R(o_1)(p) = q\}\right)\right) \\
& + \sum_{\substack{o_1 \in \mathcal{B}_0^{<i}(R) \\ q \in \mathcal{P}_{\circ}^{\mathsf{f}}(B_R(o_1))}} a_{B_R(o_1)}(q) \\
& + \sum_{o_1 \in \mathcal{B}_0^{\geq i}(R)} \left(\mathsf{Card}\left(e(o_1)\right) \cdot \mathsf{Card}\left(\{p \in \mathcal{P}_{\bullet}^{\mathsf{f}}(B_R(o_1)); b_R(o_1)(p) = q\}\right)\right) \\
& + \sum_{\substack{o_1 \in \mathcal{B}_0^{=i}(R) \\ q \in \mathcal{P}_{\circ}^{\mathsf{f}}(B_R(o_1))}} \left(\mathsf{Card}\left(e(o_1)\right) \cdot a_{B_R(o_1)}(q)\right) \\
& + \sum_{\substack{o_1 \in \mathcal{B}_0^{\geq i+1}(R) \\ q \in \mathcal{P}_{\circ}^{\mathsf{f}}(B_R(o_1))}} \left(a_{B_R(o_1),i}(q) \cdot \mathsf{Card}\left(e(o_1)\right)\right) + \sum_{e_1 \in e(o_1)} \sum_{o \in \mathcal{B}_{B_R(o_1)}^{\geq i}(q)} \sum_{\substack{j \in \mathbb{N} \\ k^j \in e_1^{\#}(o)}} m_{k,j}(\mathcal{T}(e_1)[i])(q) \cdot k^j
\end{aligned}
$$

(by Lemma 15 and by induction hypothesis)

$\square$

B.2 Proof of Proposition 5

Proof. (Sketch) The proof is in four steps, using Lemmas 4 and 13.

1. We first prove that, for any $j_0 \in \mathbb{N} \setminus \{0\}$, for any $o_1 \in \mathcal{B}_0^{\geq i}(R)$, for any $e_1 \in e(o_1)$, we have $\mathcal{K}_{k,j_0}(\mathcal{T}(e_1)[i]) \cap \mathcal{P}_{\circ}^{\mathsf{f}}(B_R(o_1)) \subseteq \mathcal{K}_{k,j_0}(\mathcal{T}(e)[i])$.

2. Second we prove, by induction on $depth(R)$, that

- for any $j \in \mathcal{N}_i(e)$, there exists $o \in \mathcal{B}^{=i}(R)$ such that $k^j \in e^\#(o)$ and

  - either $o \in \mathcal{B}_0^{=i}(R)$; moreover, in this case, $e^\#(o) = \{k^j\}$ and $\mathcal{K}_{k,j}(\mathcal{T}(e)[i]) = \mathsf{im}(b_R(o))$;
  - or there exist $o_1 \in \mathcal{B}_0^{\geq i+1}(R)$ and $o' \in \mathcal{B}^{=i}(B_R(o_1))$ such that $o = o_1 : o'$; moreover, in this case, there exists $e_1 \in e(o_1)$ such that $j \in \mathcal{N}_i(e_1)$ and $\mathcal{K}_{k,j}(\mathcal{T}(e)[i]) = \{(o_1, e_1) : p;$ $p \in \mathcal{K}_{k,j}(\mathcal{T}(e_1)[i]) \setminus \mathcal{P}_\circ^f(B_R(o_1))\} \cup (\mathcal{K}_{k,j}(\mathcal{T}(e_1)[i]) \cap \mathcal{P}_\circ^f(B_R(o_1))) \cup \{b_R(o_1)(q); q \in \mathcal{P} \cap \mathcal{P}^f(B_R(o_1)) \text{ and } a_{\mathcal{T}(e_1)[i]}(q) < k\}$;

- and, for any $j > 0$ such that $\mathcal{K}_{k,j}(\mathcal{T}(e)[i]) \neq \emptyset$, we have $j \in \mathcal{M}_i(e)$.

3. We prove, by induction on $depth(R)$, that, for any $o \in \mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1])$, there exists $j \in \mathcal{N}_i(e)$ such that $\mathsf{im}(b_{\mathcal{T}(e)[i+1]}(o)) = \mathcal{K}_{k,j}(\mathcal{T}(e)[i])$.

4. Finally we prove, by induction on $depth(R)$, that, for any $j \in \mathcal{N}_i(e)$, we have $!_{e,i}(j) \in \mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1])$.

$\square$

C Rebuilding the boxes: Proposition 16

Proposition 16. *Let R be a $\circ$-PS. Let $k > \mathsf{Card}(\mathcal{B}(R)), cosize(R)$. Let e be a k-injective pseudo-experiment of R. Let $i \in \mathbb{N}$. Let $j_0 \in \mathcal{N}_i(e)$. We set $\mathcal{T} = \mathcal{S}_{\mathcal{T}(e)[i]}^k(\mathcal{K}_{k,j_0}(\mathcal{T}(e)[i]))$. For any $T \in \mathcal{T}$, let $(m_j^T)_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^\mathbb{N}$ such that $\mathsf{Card}(\{T' \in \mathcal{T}; T' \equiv T\}) = \sum_{j \in \mathbb{N}} m_j^T \cdot k^j$. Let $\mathcal{U} \subseteq \mathcal{T}$ such that, for any $T \in \mathcal{U}$, $\mathsf{Card}(\{T' \in \mathcal{U}; T \equiv T'\}) = m_{j_0}^T$. Then there exists $\rho : B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0)) \simeq \sum \overline{\mathcal{U}}$ such that $b_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))(q) =$*
$$\begin{cases} \mathcal{G}(\rho)(q) & \text{if } q \in \mathcal{P}_\circ^f(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))); \\ t_{\mathcal{G}(\mathcal{T}(e)[i])}(\mathcal{G}(\rho)(q)) & \text{if } q \in \mathcal{P}_\bullet^f(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))). \end{cases}$$

The proof of Proposition 16 uses Proposition 19, which justifies that in the algorithm leading from $\mathcal{T}(e)[i]$ to $\mathcal{T}(e)[i+1]$, for every $j_0 \in \mathcal{N}_i(e)$, for every equivalence class $\mathfrak{T} \in \mathcal{S}_{\mathcal{T}(e)[i]}^k(\mathcal{K}_{k,j_0}(\mathcal{T}(e)[i]))_{/\equiv}$, if $\mathsf{Card}(\mathfrak{T}) = \sum_{j \in \mathbb{N}} m_j \cdot k^j$ (with $0 \leq m_j < k$), then we remove $m_{j_0} \cdot k^{j_0}$ elements of $\mathfrak{T}$ from $\mathcal{G}(\mathcal{T}(e)[i])$. Its proof is quite technical, we just state the right induction hypothesis and give some remarks.

Definition 19. *Let R be a $\circ$-PS. Let $o_1 \in \mathcal{B}_0^{\geq i}(R)$. Let $k > 1$. Let e be a k-injective pseudo-experiment of R and let $e_1 \in e(o_1)$. Let $i \in \mathbb{N}$. Let $\mathcal{P} \subseteq \mathcal{P}_0(\mathcal{T}(e_1)[i])$. We denote by $\mathcal{P}[R, o_1, e_1, i]_k$ the following subset of $\mathcal{P}_0(\mathcal{T}(e)[i])$:*

$$\begin{aligned} &\mathcal{P}[R, o_1, e_1, i]_k \\ =\ & \{(o_1, e_1) : p; p \in \mathcal{P} \setminus \mathcal{P}_\circ^f(B_R(o_1))\} \cup (\mathcal{P} \cap \mathcal{P}_\circ^f(B_R(o_1))) \\ & \cup \{b_R(o_1)(q); q \in \mathcal{P} \cap \mathcal{P}^f(B_R(o_1)) \text{ and } a_{\mathcal{T}(e_1)[i]}(q) < k\} \end{aligned}$$

Fact 17. *Let S be a differential $\circ$-PS. Let $\mathcal{P} \subseteq \mathcal{P}_0(S)$ and let $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$ such that*

1. $(\forall w \in \mathcal{W}_0(S) \setminus \mathcal{P})(t_{\mathcal{G}(S)}(w) \in \mathcal{P} \Rightarrow t_{\mathcal{G}(S)}(w) \in \mathcal{P}_0^e(S))$

2. $(\forall w \in \mathcal{W}_0(S) \cap \mathcal{Q})(t_{\mathcal{G}(S)}(w) \in \mathcal{P} \Rightarrow t_{\mathcal{G}(S)}(w) \in \mathcal{Q})$

3. *and* $(\forall o_1 \in \mathcal{B}_0(S) \cap \mathcal{P})\mathsf{im}(b_S(o_1)) \subseteq \mathcal{P}$.

Then there exists a substructure R of S such that $\mathcal{P}_0(R) = \mathcal{P}$ and $R \sqsubseteq_\mathcal{Q} S$.

Proof. We set $\mathcal{G} = (\mathcal{W}, \mathcal{P}, l, t, \mathcal{L}, \mathsf{T})$ and $R = (\mathcal{G}, \mathcal{B}_0, B, b)$ with:

- $\mathcal{W} = \{w \in \mathcal{W}_0(S) \cap (\mathcal{P} \setminus \mathcal{Q}); t_{\mathcal{G}(S)}(w) \in \mathcal{P}\}$

- $l = l_{\mathcal{G}(S)}|_\mathcal{P}$

- $t = t_{\mathcal{G}(S)}|_\mathcal{W}$

- $\mathcal{L} = \mathcal{L}(\mathcal{G}(S)) \cap \{w \in \mathcal{W}_0(S); t_{\mathcal{W}_0(S)}(w) \in \mathcal{P}\}$

- $\mathsf{T} = \mathsf{T}_{\mathcal{G}(S)}|_{\mathcal{P}}$

- $\mathcal{B}_0 = \mathcal{B}_0(S) \cap \mathcal{P}$

- $B = B_S|_{\mathcal{B}_0}$

- and $b = b_S|_{\mathcal{B}}$.

$\square$

If the pair $(\mathcal{P}, \mathcal{Q})$ satisfies the conditions of the previous fact, then we say that the pair $(\mathcal{P}, \mathcal{Q})$ is *adequate* with respect to S and we denote by $S|_{(\mathcal{P},\mathcal{Q})}$ the unique substructure R of S such that $\mathcal{P}_0(R) = \mathcal{P}$ and $R \sqsubseteq_{\mathcal{Q}} S$. We have $S|_{\mathcal{P}} = S|_{(\mathcal{P},\emptyset)}$.

Fact 18. *Let R be a $\circ$-PS. Let $k > 1$. Let e be a k-injective pseudo-experiment of R. Let $i \in \mathbb{N}$. Let $o_1 \in \mathcal{B}_0^{\geq i}(R)$. Let $e_1 \in e(o_1)$. Let $\mathcal{P} \subseteq \mathcal{P}_0(\mathcal{T}(e_1)[i])$ and $\mathcal{Q} \subseteq \mathcal{P}_0^e(\mathcal{T}(e_1)[i])$ such that the pair $(\mathcal{P}, \mathcal{Q})$ is adequate with respect to $\mathcal{T}(e_1)[i]$. Then the pair $(\mathcal{P}[R, o_1, e_1, i]_k, \mathcal{Q}[R, o_1, e_1, i]_k)$ is adequate with respect to $\mathcal{T}(e)[i]$. Moreover we have*

- $\mathcal{P}^f(\mathcal{T}(e)[i]|_{\mathcal{P}[R,o_1,e_1,i]_k}) \subseteq im(b_R(o_1)) \cup \{(o_1, e_1) : p; p \in \mathcal{P}^f(\mathcal{T}(e_1)[i]|_{\mathcal{P}}) \setminus \mathcal{P}^f(B_R(o_1))\} \cup \{(o_1, e_1) : p; p \in \mathcal{P}^f(\mathcal{T}(e_1)[i]|_{\mathcal{P}}) \cap \mathcal{P}_\bullet^f(B_R(o_1))$ and $a_{\mathcal{T}(e_1)[i]}(p) \geq k\}$;

- $\mathcal{B}_0(\mathcal{T}(e)[i]|_{\mathcal{P}[R,o_1,e_1,i]_k}) = \{(o_1, e_1) : o; o \in \mathcal{B}_0(\mathcal{T}(e_1)[i]|_{\mathcal{P}})\}$;

- *and, for any $p \in \mathcal{P} \setminus \mathcal{P}_\circ^f(B_R(o_1))$, $a_T((o_1, e_1) : p) = a_{\mathcal{T}(e_1)[i]|_{\mathcal{P}}}(p)$.*

This fact allows the following definition:

Definition 20. *Let R be a $\circ$-PS. Let e be a pseudo-experiment of R. Let $i, k \in \mathbb{N}$. Let $o_1 \in \mathcal{B}_0^{\geq i}(R)$. Let $e_1 \in e(o_1)$. Let $T \sqsubseteq \mathcal{T}(e_1)[i]$. We set $T[R, e, i, o_1, e_1]_k = \mathcal{T}(e)[i]|_{(\mathcal{P}_0(T)[R,o_1,e_1,i]_k, \mathcal{P}^f(T)[R,o_1,e_1,i]_k)}$.*

Before of the proof of Proposition 19, notice that the proof of Proposition 5 actually proves something more than its statement: it proves also that, for any $j \in \mathcal{N}_i(e)$ such that $!_{e,i}(j) \notin \mathcal{B}_0^{=i}(R)$, there exist $o_1 \in \mathcal{B}_0^{\geq i+1}(R)$ and $e_1 \in e(o_1)$ such that $j \in \mathcal{N}_i(e_1)$ and $\mathcal{K}_{k,j}(\mathcal{T}(e)[i]) = \mathcal{K}_{k,j}(\mathcal{T}(e_1)[i])[R, o_1, e_1, i]_k$. From now on, whenever we refer to Proposition 5, we refer to the statement thus completed.

Proposition 19. *Let R be a $\circ$-PS. Let $k > \mathsf{Card}(\mathcal{B}(R)), cosize(R)$. Let e be a k-injective pseudo-experiment of R. Let $i \in \mathbb{N}$. Let $T \in \mathcal{S}_{\mathcal{T}(e)[i]}^k((\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}(e)[i]), \mathcal{P}^f(R)))$. We set*

$$\mathcal{T} = \{T' \in \mathcal{S}_{\mathcal{T}(e)[i]}^k((\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}(e)[i]), \mathcal{P}^f(R))); T \equiv T'\}$$

and

$$\mathcal{T}' = \{T' \in \mathcal{S}_{\mathcal{T}(e)[i+1]}^k((\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}(e)[i]), \mathcal{P}^f(R))); T \equiv T'\}$$

Let $(m_j)_{j \in \mathbb{N}}, (m_j')_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ such that $\mathsf{Card}(\mathcal{T}) = \sum_{j \in \mathbb{N}} m_j \cdot k^j$ and $\mathsf{Card}(\mathcal{T}') = \sum_{j \in \mathbb{N}} m_j' \cdot k^j$. Then

- $\{j \in \mathbb{N} \setminus \{0\}; m_j \neq 0\} \subseteq \mathcal{M}_i(e)$

- $\{j \in \mathbb{N} \setminus \{0\}; m_j' \neq 0\} \subseteq \mathcal{M}_{i+1}(e)$

- *and $(\forall j \in \mathcal{M}_{i+1}(e)) m_j' = m_j$.*

Moreover, if $\{T' \in \mathcal{T}; T' \sqsubseteq R\} \neq \emptyset$, then $m_0' = \mathsf{Card}(\{T' \in \mathcal{T}; T' \sqsubseteq R\}) = m_0$.

Proof. By induction on $depth(R)$. We distinguish between two cases:

1. Case $\mathcal{P}^{\mathsf{f}}(T)\backslash\mathcal{P}_0(R) \neq \emptyset$: by Proposition 5, there exist $o_1 \in \mathcal{B}_0^{\geq i+1}(R)$, $e_1 \in e(o_1)$ and $p' \in \mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i])\backslash$
$\mathcal{P}_{\circ}^{\mathsf{f}}(B_R(o_1))$ such that $(o_1, e_1) : p' \in \mathcal{P}^{\mathsf{f}}(T)$. For any $T' \in \mathcal{T}$, we set $\mathcal{P}_{T'} = \{q \in \mathcal{P}_0(\mathcal{T}(e_1)[i]); (o_1, e_1) : q$
$\in \mathcal{P}_0(T')\}$ and

$$U_{T'} = \mathcal{T}(e_1)[i]\big|_{(\mathcal{P}_{T'} \cup \bigcup_{o \in \mathcal{P}_{T'} \cap \mathcal{B}_0(B_R(o_1))} \mathrm{im}(b_{B_R(o_1)}(o)), \mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]) \cup \mathcal{P}^{\mathsf{f}}(B_R(o_1)))}$$

we have $T' = U_{T'}[R, e, i, o_1, e_1]_k$, hence $cosize(U_{T'}) \leq cosize(T) < k$; therefore $U_{T'} \in \mathcal{S}_{\mathcal{T}(e_1)[i]}^k((\mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]), \mathcal{P}^{\mathsf{f}}(B_R$

   • Now, notice that if there exists $q \in \mathcal{P}^{\mathsf{f}}(U_T) \cap \mathcal{P}_{\bullet}^{\mathsf{f}}(B_R(o_1))$ such that $q \notin \mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i])$, then:

$$(\forall T' \in \mathcal{T})U_{T'} \in \mathcal{S}_{B_R(o_1)}^k((\mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]), \mathcal{P}^{\mathsf{f}}(B_R(o_1))))$$

   we thus have $(\forall T' \in \mathcal{T})T' \sqsubseteq R$ and $\mathsf{Card}\,(\mathcal{T}) \leq a_R(b_R(o_1)(q)) < k$; we obtain $m_0 = \mathsf{Card}\,(\{T' \sqsubseteq R; T' \equiv T\})$
   and $\{j \in \mathbb{N} \setminus \{0\}; m_j \neq 0\} = \emptyset$. In the same way, we have

$$(\forall T' \in \mathcal{T}')U_{T'} \in \mathcal{S}_{B_R(o_1)}^k((\mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]), \mathcal{P}^{\mathsf{f}}(B_R(o_1))))$$

   we thus have $(\forall T' \in \mathcal{T}')T' \sqsubseteq R$ and $\mathsf{Card}\,(\mathcal{T}') \leq a_R(b_R(o_1)(q)) < k$; we obtain $m'_0 = \mathsf{Card}\,(\{T' \sqsubseteq R; T' \equiv T\})$
   and $\{j \in \mathbb{N} \setminus \{0\}; m'_j \neq 0\} = \emptyset$.

   • Otherwise, for any $T' \in \mathcal{T}$, we have $\mathcal{P}^{\mathsf{f}}(U_{T'}) = \mathcal{P}^{\mathsf{f}}(U_T)$, hence $U_{T'} \equiv U_T$. So,

$$\mathsf{Card}\,(\mathcal{T}) = \mathsf{Card}\,\left(\{U \in \mathcal{S}_{\mathcal{T}(e_1)[i]}^k((\mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]), \mathcal{P}^{\mathsf{f}}(B_R(o_1)))); U \equiv U_T\}\right)$$

   and
$$\mathsf{Card}\,(\mathcal{T}') = \mathsf{Card}\,\left(\{U \in \mathcal{S}_{\mathcal{T}(e_1)[i+1]}^k((\mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]), \mathcal{P}^{\mathsf{f}}(B_R(o_1)))); U \equiv U_T\}\right)$$

   We apply the induction hypothesis and we obtain:

   – $\{j \in \mathbb{N} \setminus \{0\}; m_j \neq 0\} \subseteq \mathcal{M}_i(e_1) \subseteq \mathcal{M}_i(e)$
   – $\{j \in \mathbb{N} \setminus \{0\}; m'_j \neq 0\} \subseteq \mathcal{M}_{i+1}(e_1) \subseteq \mathcal{M}_{i+1}(e)$
   – and $(\forall j \in \mathcal{M}_{i+1}(e_1))m'_j = m_j$.

2. Case $\mathcal{P}^{\mathsf{f}}(T) \setminus \mathcal{P}_0(R) = \emptyset$: Let $o_1 \in \mathcal{B}_0^{\geq i}(R)$. We set $\mathcal{T}_{o_1} = \{U \in \mathcal{C}^k(B_R(o_1)); U[R, e, i, o_1, e_1]_k \equiv T \text{ and } \mathcal{P}_{\bullet}^{\mathsf{f}}(U) \neq \emptyset\}$. We set $m^{o_1} = \mathsf{Card}\,(\mathcal{T}_{o_1})$. We have $m^{o_1} < k$. Let $e_1 \in e(o_1)$.

   Notice that, for any $U \in \mathcal{S}_{\mathcal{T}(e_1)[i]}^k((\mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]), \mathcal{P}^{\mathsf{f}}(B_R(o_1))))$ such that $U[R, e, i, o_1, e_1]_k \equiv T$,
   we have $U \in \mathcal{C}^k(\mathcal{T}(e_1)[i])$.

   Now, for any $U \in \mathcal{C}^k(\mathcal{T}(e_1)[i])$ such that $\mathcal{P}_{\bullet}^{\mathsf{f}}(U) \neq \emptyset$, we have $U \sqsubseteq B_R(o_1)$. Indeed assume that $U \in$
   $\mathcal{C}^k(\mathcal{T}(e_1)[i])$ such that $\mathcal{P}_{\bullet}^{\mathsf{f}}(U) \neq \emptyset$ and $\mathcal{P}_0(U) \setminus \mathcal{P}_0(B_R(o_1)) \neq \emptyset$; let $q \in \mathcal{P}_0(U) \setminus \mathcal{P}_0(B_R(o_1))$ and let $q' \in$
   $\mathcal{P}_{\bullet}^{\mathsf{f}}(U)$; there exist $p_0, \ldots p_{n+1} \in \mathcal{P}_0(U)$ such that $p_0 = q'$, $p_{n+1} = q$, $(\forall j \in \{0, \ldots, n\})p_j \frown_{\mathcal{T}(e_1)[i]} p_{j+1}$
   and $(\forall j \in \{0, \ldots, n\})(p_j \in \mathcal{P}_{\circ}^{\mathsf{f}}(U) \Rightarrow j \in \{0, n+1\})$. Let $j_0 = \max\{j \in \{0, \ldots, n\}; p_j \in \mathcal{P}_0(B_R(o_1))\}$.
   We have $a_{B_R(o_1)}(p_{j_0}) = a_{\mathcal{T}(e_1)[i]}(p_{j_0}) \geq k > cosize(B_R(o_1))$, which is contradictory.

   We thus obtain $\mathsf{Card}\,(\{U \in \mathcal{C}^k(\mathcal{T}(e_1)[i]); \mathcal{P}_{\bullet}^{\mathsf{f}}(U) \neq \emptyset \text{ and } U[R, e, i, o_1, e_1]_k \equiv T\}) = m^{o_1}$. We distinguish between two cases:

   • $\{U \in \mathcal{C}^k(\mathcal{T}(e_1)[i]); \mathcal{P}_{\bullet}^{\mathsf{f}}(U) = \emptyset \text{ and } U[R, e, i, o_1, e_1]_k \equiv T\} = \emptyset$: for any $j \in \mathcal{M}_i(e_1)$, we set $m_j^{(o_1,e_1)} = 0$.

   • $\{U \in \mathcal{C}^k(\mathcal{T}(e_1)[i]); \mathcal{P}_{\bullet}^{\mathsf{f}}(U) = \emptyset \text{ and } U[R, e, i, o_1, e_1]_k \equiv T\} \neq \emptyset$: let $U \in \mathcal{C}^k(\mathcal{T}(e_1)[i])$ such that
   $\mathcal{P}_{\bullet}^{\mathsf{f}}(U) = \emptyset$ and $U[R, e, i, o_1, e_1]_k \equiv T$; let $(m_j^{(o_1,e_1)})_{j \in \mathbb{N}}, (m'_j{}^{(o_1,e_1)})_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ such that

$$\mathsf{Card}\,\left(\{U' \in \mathcal{S}_{\mathcal{T}(e_1)[i]}^k((\mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]), \mathcal{P}^{\mathsf{f}}(B_R(o_1)))); U' \equiv U\}\right) = \sum_{j \in \mathbb{N}} m_j^{(o_1,e_1)} \cdot k^j$$

and

$$\mathsf{Card}\left(\{U' \in \mathcal{S}^k_{\mathcal{T}(e_1)[i+1]}((\mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]), \mathcal{P}^{\mathsf{f}}(B_R(o_1)))); U' \equiv U\}\right) = \sum_{j \in \mathbb{N}} m'^{(o_1,e_1)}_j \cdot k^j$$

By induction hypothesis, we have

- $\{j \in \mathbb{N} \setminus \{0\}; m^{(o_1,e_1)}_j \neq 0\} \subseteq \mathcal{M}_i(e_1)$
- $\{j \in \mathbb{N} \setminus \{0\}; m'^{(o_1,e_1)}_j \neq 0\} \subseteq \mathcal{M}_{i+1}(e_1)$
- and $(\forall j \in \mathcal{M}_{i+1}(e_1)) m'^{(o_1,e_1)}_j = m^{(o_1,e_1)}_j$.

For any $U \in \mathcal{C}^k(\mathcal{T}(e_1)[i])$ such that $\mathcal{P}^{\mathsf{f}}_\bullet(U) = \emptyset$, we have $\neg U \sqsubseteq B_R(o_1)$, hence $m'^{(o_1,e_1)}_0 = 0 = m^{(o_1,e_1)}_0$ and $m^{o_1} = \mathsf{Card}(\{T' \in \mathcal{T}; T' \sqsubseteq R\})$.

Finally:

$$\mathsf{Card}(\{T' \in \mathcal{T}; \neg T' \sqsubseteq R\})$$

$$= \mathsf{Card}\left(\bigcup_{o_1 \in \mathcal{B}^{\geq i}_0(R)} \bigcup_{e_1 \in e(o_1)} \left\{U' \in \mathcal{S}^k_{\mathcal{T}(e_1)[i]}((\mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]), \mathcal{P}^{\mathsf{f}}(B_R(o_1)))); U'[R,e,i,o_1,e_1]_k \in \mathcal{T}\right\}\right)$$

$$= \mathsf{Card}\left(\bigcup_{o_1 \in \mathcal{B}^{\geq i}_0(R)} \bigcup_{e_1 \in e(o_1)} \{U' \in \mathcal{C}^k(\mathcal{T}(e_1)[i]); U'[R,e,i,o_1,e_1]_k \in \mathcal{T}\}\right)$$

$$= \sum_{o_1 \in \mathcal{B}^{\geq i}_0(R)} \sum_{e_1 \in e(o_1)} \mathsf{Card}\left(\{U' \in \mathcal{C}^k(\mathcal{T}(e_1)[i]); U'[R,e,i,o_1,e_1]_k \in \mathcal{T}\}\right)$$

$$= \sum_{o_1 \in \mathcal{B}^{\geq i}_0(R)} \sum_{e_1 \in e(o_1)} \left(m^{o_1} + \mathsf{Card}\left(\{U' \in \mathcal{C}^k(\mathcal{T}(e_1)[i]); U'[R,e,i,o_1,e_1]_k \in \mathcal{T} \text{ and } \mathcal{P}^{\mathsf{f}}_\bullet(U') = \emptyset\}\right)\right)$$

$$= \sum_{o_1 \in \mathcal{B}^{\geq i}_0(R)} \sum_{e_1 \in e(o_1)} \left(m^{o_1} + \sum_{j \in \mathcal{M}_i(e_1)} m^{(o_1,e_1)}_j \cdot k^j\right)$$

$$= \sum_{o_1 \in \mathcal{B}^{\geq i}_0(R)} \left(m^{o_1} \cdot \mathsf{Card}(e(o_1)) + \sum_{e_1 \in e(o_1)} \sum_{j \in \mathcal{M}_i(e_1)} m^{(o_1,e_1)}_j \cdot k^j\right)$$

and

$$\mathsf{Card}\left(\{T' \in \mathcal{T}'; \neg T' \sqsubseteq R\}\right)$$

$$= \ \mathsf{Card}\left(\bigcup_{o_1 \in \mathcal{B}_0^{\geq i+1}(R)} \bigcup_{e_1 \in e(o_1)} \left\{U' \in \mathcal{S}_{\mathcal{T}(e_1)[i+1]}^k((\mathcal{K}_{k,\mathcal{N}_i(e_1)}(\mathcal{T}(e_1)[i]), \mathcal{P}^{\mathsf{f}}(B_R(o_1)))); U'[R,e,i,o_1,e_1]_k \in \mathcal{T}'\right\}\right)$$

$$= \ \mathsf{Card}\left(\bigcup_{o_1 \in \mathcal{B}_0^{\geq i+1}(R)} \bigcup_{e_1 \in e(o_1)} \left\{U' \in \mathcal{C}^k(\mathcal{T}(e_1)[i+1]); U'[R,e,i,o_1,e_1]_k \in \mathcal{T}'\right\}\right)$$

$$= \ \sum_{o_1 \in \mathcal{B}_0^{\geq i+1}(R)} \sum_{e_1 \in e(o_1)} \mathsf{Card}\left(\left\{U' \in \mathcal{C}^k(\mathcal{T}(e_1)[i+1]); U'[R,e,i,o_1,e_1]_k \in \mathcal{T}'\right\}\right)$$

$$= \ \sum_{o_1 \in \mathcal{B}_0^{\geq i+1}(R)} \sum_{e_1 \in e(o_1)} \left(m^{o_1} + \mathsf{Card}\left(\left\{U' \in \mathcal{C}^k(\mathcal{T}(e_1)[i+1]); U'[R,e,i,o_1,e_1]_k \in \mathcal{T}' \ and \ \mathcal{P}_\bullet^{\mathsf{f}}(U') = \emptyset\right\}\right)\right)$$

$$= \ \sum_{o_1 \in \mathcal{B}_0^{\geq i+1}(R)} \sum_{e_1 \in e(o_1)} \left(m^{o_1} + \sum_{j \in \mathcal{M}_{i+1}(e_1)} m_j^{(o_1,e_1)} \cdot k^j\right)$$

$$= \ \sum_{o_1 \in \mathcal{B}_0^{\geq i+1}(R)} \left(m^{o_1} \cdot \mathsf{Card}\left(e(o_1)\right) + \sum_{e_1 \in e(o_1)} \sum_{j \in \mathcal{M}_{i+1}(e_1)} m_j^{(o_1,e_1)} \cdot k^j\right)$$

Therefore $\{j \in \mathbb{N} \setminus \{0\}; m_j \neq 0\} \subseteq \bigcup_{o_1 \in \mathcal{B}_0^{\geq i}(R)}(\{\log_k(\mathsf{Card}\,(e(o_1))\} \cup \bigcup_{e_1 \in e(o_1)} \mathcal{M}_i(e_1)) \subseteq \mathcal{M}_i(e)$ and $\{j \in \mathbb{N}\setminus\{0\}; m'_j \neq 0\} \subseteq \bigcup_{o_1 \in \mathcal{B}_0^{\geq i+1}(R)}(\{\log_k(\mathsf{Card}\,(e(o_1))\} \cup \bigcup_{e_1 \in e(o_1)} \mathcal{M}_{i+1}(e_1)) \subseteq \mathcal{M}_{i+1}(e)$. Moreover $m_0 = m'_0 = \mathsf{Card}\left(\{T' \in \mathcal{T}; T' \sqsubseteq R\}\right)$. Lastly, for any $j \in \mathcal{M}_{i+1}(e)$, we have $m'_j = \sum_{e_1 \in e(o_1)} \sum_{j \in \mathcal{M}_i(e_1)} m_j^{(o_1,e_1)}$. $\qquad\square$

We need a variant of the notion of equivalence denoted by $\equiv$:

Definition 21. *Let R be a $\circ$-PS. Let $k > 1$. For any k-injective pseudo-experiment e of R, for any $i \in \mathbb{N}$, for any $o \in \mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1])$, for any $T' \in \mathcal{C}^k(B_{\mathcal{T}(e)[i+1]}(o))$, for any $\circ$-PS T'' such that $\mathcal{P}_\bullet^{\mathsf{f}}(T'') \subseteq \mathcal{W}_0(\mathcal{T}(e)[i])$, we write $\varphi : T' \equiv_{e,i,o} T''$ if $\varphi : T' \simeq T''$ such that $(\forall q \in \mathcal{P}_\circ^{\mathsf{f}}(T'))\mathcal{G}(\varphi)(q) = b_{\mathcal{T}(e)[i+1]}(o)(q)$ and $(\forall q \in \mathcal{P}_\bullet^{\mathsf{f}}(T'))t_{\mathcal{G}(\mathcal{T}(e)[i])}(\mathcal{G}(\varphi)(q)) = b_{\mathcal{T}(e)[i+1]}(o)(q)$ and $T' \equiv_{e,i,o} T''$ if there exists φ such that $\varphi : T' \equiv_{e,i,o} T''$.*

Sketch of the proof of Proposition 16

Proof. (Sketch) We set $\mathcal{W} = \mathcal{C}^k(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0)))$. Let $\tau : \mathcal{W}_{/\equiv} \to \mathcal{W}$ such that, for any $\mathcal{V} \in \mathcal{W}_{/\equiv}$, we have $\tau(\mathcal{V}) \in \mathcal{V}$. For any $\mathcal{V} \in \mathcal{W}_{/\equiv}$, there exist $\sigma(\mathcal{V}) \in \mathcal{T}$ and $\varphi_\mathcal{V} : \tau(\mathcal{V}) \simeq \overline{\sigma(\mathcal{V})}$ such that, for any $q \in \mathcal{P}^{\mathsf{f}}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))) \cap \mathcal{P}_0(\tau(\mathcal{V}))$, we have $b_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))(q) = \begin{cases} \mathcal{G}(\varphi_\mathcal{V})(q) & if \ q \in \mathcal{P}_\circ^{\mathsf{f}}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))); \\ t_{\mathcal{G}(\mathcal{T}(e)[i])}(\mathcal{G}(\varphi_\mathcal{V})(q)) & if \ q \in \mathcal{P}_\bullet^{\mathsf{f}}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))). \end{cases}$
We check the existence of such $\sigma(\mathcal{V})$ and $\varphi(\mathcal{V})$ by induction on $depth(R)$:

- In the case $!_{e,i}(j_0) \in \mathcal{B}_0^{=i}(R)$, for any $e_1 \in e(!_{e,i}(j_0))$, we have $\tau(\mathcal{V}) \in \mathcal{C}^k(\mathcal{T}(e_1)[i])$, hence one can set $\sigma(\mathcal{V}) = \tau(\mathcal{V})[R, e, i, !_{e,i}(j_0), e_1]_k$ for some $e_1 \in e(!_{e,i}(j_0))$.

- In the case $!_{e,i}(j_0) = (o_1, e_1) \ :!_{e_1,i}(j_0)$ for some $o_1 \in \mathcal{B}_0^{\geq i+1}(R)$ and some $e_1 \in e(o_1)$, by induction hypothesis, there exists $\sigma'(\mathcal{V}) \in \mathcal{S}_{\mathcal{T}(e_1)[i]}^k(\mathcal{K}_{k,j_0}(\mathcal{T}(e_1)[i]))$ and $\varphi'_\mathcal{V} : \tau(\mathcal{V}) \simeq \overline{\sigma'(\mathcal{V})}$ such that, for any $q \in \mathcal{P}^{\mathsf{f}}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))) \cap \mathcal{P}_0(\tau(\mathcal{V}))$, we have $b_{\mathcal{T}(e_1)[i+1]}(!_{e_1,i}(j_0))(q) = \begin{cases} \mathcal{G}(\varphi'_\mathcal{V})(q) & if \ q \in \mathcal{P}_\circ^{\mathsf{f}}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0 \\ t_{\mathcal{G}(\mathcal{T}(e_1)[i])}(\mathcal{G}(\varphi'_\mathcal{V})(q)) & if \ q \in \mathcal{P}_\bullet^{\mathsf{f}}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0 \end{cases}$

  We set $\sigma(\mathcal{V}) = \sigma'(\mathcal{V})[R, e, i, !_{e,i}(j_0), e_1]_k$. Moreover we set $(\varphi_\mathcal{V})_\mathcal{G} : \begin{array}{ccc} \mathcal{P}_0(\tau(\mathcal{V})) & \to & \mathcal{P}_0(\overline{\sigma(\mathcal{V})}) \\ p & \mapsto & \begin{cases} \mathcal{G}(\varphi'_\mathcal{V})(p) & if \ \mathcal{G}(\varphi'_\mathcal{V})(p) \in \\ (o_1, e_1) : \mathcal{G}(\varphi'_\mathcal{V})(p) & otherwise; \end{cases} \end{array}$

  and $\varphi(\mathcal{V}) = ((\varphi_\mathcal{V})_\mathcal{G}, (\varphi'(\mathcal{V})(o))_{o \in \mathcal{B}_0(\tau(\mathcal{V}))})$.

26

We set $\mathcal{U}_{\mathcal{V}} = \{T' \in \mathcal{U}; T' \equiv \sigma(\mathcal{V})\}$.

Let us show that there exists a bijection $\epsilon_{\mathcal{V}} : \mathcal{V} \to \mathcal{U}_{\mathcal{V}}$: there exist a partition $\mathfrak{U}$ of $\{U' \in \mathcal{T} \setminus \mathcal{S}^k_{\mathcal{T}(e)[i+1]}(\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}(e)[i])); U' \equiv \sigma(\mathcal{V})\}$ and a bijection $\delta : \{!_{e,i}(j); j \in \mathcal{N}_i(e) \ and \ n_j \neq 0\} \to \mathfrak{U}$ such that, for any $j \in \mathcal{N}_i(e)$ such that $n_j \neq 0$, we have $\mathsf{Card}\,(\delta(!_{e,i}(j))) = n_j \cdot k^j$, where n_j is the integer $\mathsf{Card}\left(\{T' \in \mathcal{W}; T' \equiv_{e,i,o} \overline{\sigma(\mathcal{V})}\}\right)$. We have $\mathsf{Card}\,(\bigcup \mathfrak{U}) = \sum_{j \in \mathcal{N}_i(e)} n_j \cdot k^j$. By Proposition 19, we have $\mathsf{Card}\,(\bigcup \mathfrak{U}) = \sum_{j \in \mathcal{N}_i(e)} m_j^{\sigma(\mathcal{V})} \cdot k^j$. We thus have $(\forall j \in \mathcal{N}_i(e)) n_j = m_j^{\sigma(\mathcal{V})}$.

For any $V \in \mathcal{V}$, there exists $\rho_V : V \simeq \overline{\epsilon_{\mathcal{V}}(V)}$ such that, for any $q \in \mathcal{P}^{\mathsf{f}}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))) \cap \mathcal{P}_0(V)$,

$b_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))(q) = \begin{cases} \mathcal{G}(\rho_V)(q) & \text{if } q \in \mathcal{P}^f_{\circ}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))); \\ t_{\mathcal{G}(\mathcal{T}(e)[i])}(\mathcal{G}(\rho_V)(q)) & \text{if } q \in \mathcal{P}^f_{\bullet}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))). \end{cases}$ We have: for any $\mathcal{V} \in \mathcal{W}_{/\equiv}$,

there exists $\rho_{\mathcal{V}} : \sum_{V \in \mathcal{V}} V \simeq \sum_{V \in \mathcal{V}} \overline{\epsilon_{\mathcal{V}}(V)}$ such that, for any $V_1, V_2 \in \mathcal{V}$, $\mathcal{G}(\rho_{V_1})|_{\mathcal{P}^f_{\circ}(V_1) \cap \mathcal{P}^f_{\circ}(V_2)} = \mathcal{G}(\rho_{\mathcal{V}})|_{\mathcal{P}^f_{\circ}(V_1) \cap \mathcal{P}^f_{\circ}(V_2)} = \mathcal{G}(\rho_{V_2})|_{\mathcal{P}^f_{\circ}(V_1) \cap \mathcal{P}^f_{\circ}(V_2)}$; the sets $\{\sum_{V \in \mathcal{V}} V; \mathcal{V} \in \mathcal{W}_{/\equiv}\}$ and $\{\sum_{V \in \mathcal{V}} \overline{\epsilon_{\mathcal{V}}(V)}; \mathcal{V} \in \mathcal{W}_{/\equiv}\}$ are gluable; there exists $\rho : \sum_{\mathcal{V} \in \mathcal{W}_{/\equiv}} \sum_{V \in \mathcal{V}} V \simeq \sum_{\mathcal{V} \in \mathcal{W}_{/\equiv}} \sum_{V \in \mathcal{V}} \overline{\epsilon_{\mathcal{V}}(V)}$ such that, for any $q \in \mathcal{P}^{\mathsf{f}}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0)))$, we have

$b_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))(q) = \begin{cases} \mathcal{G}(\rho)(q) & \text{if } q \in \mathcal{P}^f_{\circ}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))); \\ t_{\mathcal{G}(\mathcal{T}(e)[i])}(\mathcal{G}(\rho)(q)) & \text{if } q \in \mathcal{P}^f_{\bullet}(B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0))). \end{cases}$ But we have

$$\sum_{\mathcal{V} \in \mathcal{W}_{/\equiv}} \sum_{V \in \mathcal{V}} V = \sum \mathcal{W}$$
$$= B_{\mathcal{T}(e)[i+1]}(!_{e,i}(j_0)) (by \ Fact \ 7)$$

and $\sum_{\mathcal{V} \in \mathcal{W}_{/\equiv}} \sum_{V \in \mathcal{V}} \overline{\epsilon_{\mathcal{V}}(V)} = \sum_{\mathcal{V} \in \mathcal{W}_{/\equiv}} \sum \overline{\mathcal{U}_{\mathcal{V}}} = \sum \overline{\mathcal{U}}$. □

D Proposition 8

In the sketch of the proof of Proposition 8, we gave the complete formalized algorithm leading from $\mathcal{T}(e)[i]$ to $\mathcal{T}(e)[i+1]$ (up to the names of the ports). We only gave some arguments in favour of its correctness. In Section C, we already stated Proposition 19. This proposition is actually used in the proof of Proposition 16 and again in Proposition 8. Subsections D.1, D.2 and D.3 give some more arguments in favour of the correctness of the algorithm.

D.1 The algorithm is deterministic

We should show that, if S'' is some differential PS that enjoys the conditions given on S' in the proof of Proposition 8, then $S'' \equiv S'$. The proof is quite easy but tedious.

D.2 $\mathcal{G}(\mathcal{T}(e)[i]|_{)}\mathcal{P} \sqsubseteq \mathcal{G}(\mathcal{T}(e)[i+1])$

As said in its informal description before the statement of the proposition, in order to obtain the differential PS $\mathcal{T}(e)[i+1]$, we consider the set $\bigcup_{j \in \mathcal{N}_i(e)} \mathcal{S}^k_{\mathcal{T}(e)[i]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i]))$ and we remove some of these elements and put some other of these elements inside new boxes. This means in particular that we keep all the ports that are not caught by some element of $\bigcup_{j \in \mathcal{N}_i(e)} \mathcal{S}^k_{\mathcal{T}(e)[i]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i]))$, what is formally stated by:

$$\mathcal{G}(\mathcal{T}(e)[i]|_{)}\mathcal{P} \sqsubseteq \mathcal{G}(\mathcal{T}(e)[i+1])$$

with

$$\mathcal{P} = \{p \in \mathcal{P}_0(\mathcal{T}(e)[i+1]); p \notin \bigcup_{j \in \mathcal{N}_i(e)} \bigcup_{T \in \mathcal{S}^k_{\mathcal{T}(e)[i]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i]))} \mathcal{P}_0(T)\}$$

It is essentially what Lemma 20 says.

Lemma 20. *Let R be a $\circ$-PS. Let $k > \mathsf{Card}\,(\mathcal{B}(R))$, $cosize(R)$. Let e be a k-injective pseudo-experiment of R. Let $i \in \mathbb{N}$. For any $p \in \mathcal{P}_0(\mathcal{T}(e)[i]) \setminus \mathcal{P}_0(\mathcal{T}(e)[i+1])$, there exist $j \in \mathcal{N}_i(e)$ and $T \in \mathcal{S}^k_{\mathcal{T}(e)[i]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i]))$ such that, for any $q \in \mathcal{P}^f_\circ(T)$, we have $a_T(q) \le a_R(q)$.*

Proof. (Sketch) By induction on $depth(R)$:

If $depth(R) = 0$, then $\mathcal{P}_0(\mathcal{T}(e)[i]) \setminus \mathcal{P}_0(\mathcal{T}(e)[i+1]) = \emptyset$.

If $depth(R) > 0$, then we distinguish between two cases:

- In the case there exist $o_1 \in \mathcal{B}^{=i}_0(R)$, $e_1 \in e(o_1)$ and $p' \in \mathcal{P}_0(\mathcal{T}(e_1)[i]) \setminus \mathcal{P}^f_\circ(B_R(o_1))$ such that $p = (o_1, e_1) : p'$, there exists $T \in \mathcal{S}^k_{\mathcal{T}(e)[i]}(\operatorname{im}(b_R(o_1)))$ such that $p \in \mathcal{P}_0(T)$ and, for any $q \in \mathcal{P}^f_\circ(T)$, we have $a_T(q) \le a_R(q)$. Now, by Proposition 5, we have $\operatorname{im}(b_R(o_1)) = \mathcal{K}_{k,j}(\mathcal{T}(e)[i])$ with $e^\#(o_1) = \{k^j\}$.

- In the case there exist $o_1 \in \mathcal{B}^{\ge i+1}_0(R)$, $e_1 \in e(o_1)$ and $p' \in \mathcal{P}_0(\mathcal{T}(e_1)[i]) \setminus \mathcal{P}_0(\mathcal{T}(e_1)[i+1])$ such that $p = (o_1, e_1) : p'$, by induction hypothesis, there exists $j \in \mathcal{N}_i(e_1)$ and $T' \in \mathcal{S}^k_{\mathcal{T}(e_1)[i]}(\mathcal{K}_{k,j}(\mathcal{T}(e_1)[i]))$ such that $p' \in \mathcal{P}_0(T') \setminus \mathcal{P}^f(T')$ and, for any $q \in \mathcal{P}^f_\circ(T')$, we have $a_{T'}(q) \le a_{B_R(o_1)}(q)$.

  By Fact 18, we can set $T = T'[R, e, i, o_1, e_1]_k$. By Proposition 5, we have $T \trianglelefteq_{\mathcal{K}_{k,j}(\mathcal{T}(e)[i])} \mathcal{T}(e)[i]$.

  Let $q \in \mathcal{P}_0(T) \setminus \{(o_1, e_1) : q'; q' \in \mathcal{P}_0(T') \setminus \mathcal{P}^f_\circ(B_R(o_1))\}$. Notice that $q \in \mathcal{P}_0(R)$. We have

$$
\begin{aligned}
a_{\mathcal{G}(T)}(q) \quad \le \quad & \mathsf{Card}\,(\{w \in \mathcal{W}_0(R); t_{\mathcal{G}(R)}(w) = q\}) \\
& + \mathsf{Card}\,(\{p \in \mathcal{P}^f_\bullet(B_R(o_1)); b_R(o_1)(p) = q\})
\end{aligned}
$$

  Futhermore, notice that we have $\mathcal{B}^{<i}_0(R) \cap \mathcal{B}_0(T) = \emptyset$; so we have:

  – if $q \notin \mathcal{P}^f_\circ(B_R(o_1))$, then

$$
\begin{aligned}
& \sum_{o \in \mathcal{B}_0(T)} \mathsf{Card}\,(\{p \in \mathcal{P}^f_\bullet(B_T(o)); b_T(o)(p) = q\}) \\
& \quad + \sum_{\substack{o \in \mathcal{B}_0(T) \\ q \in \operatorname{im}(b_T(o)|_{\mathcal{P}^f_\circ(B_T(o))})}} a_{B_T(o)}(q_{T,o}) \\
= \quad & 0
\end{aligned}
$$

  – if $q \in \mathcal{P}^f_\circ(B_R(o_1))$, then

$$
\begin{aligned}
& \sum_{o \in \mathcal{B}_0(T)} \mathsf{Card}\,(\{p \in \mathcal{P}^f_\bullet(B_T(o)); b_T(o)(p) = q\}) \\
& \quad + \sum_{\substack{o \in \mathcal{B}_0(T) \\ q \in \operatorname{im}(b_T(o)|_{\mathcal{P}^f_\circ(B_T(o))})}} a_{B_T(o)}(q_{T,o}) \\
= \quad & \sum_{o' \in \mathcal{B}_0(T')} \mathsf{Card}\,(\{p \in \mathcal{P}^f_\bullet(B_{T'}(o')); b_{T'}(o')(p) = q\}) \\
& \quad + \sum_{\substack{o' \in \mathcal{B}_0(T') \\ q \in \operatorname{im}(b_{T'}(o')|_{\mathcal{P}^f_\circ(B_{T'}(o')))}}} a_{B_{T'}(o')}(q_{T',o'}) \\
\le \quad & a_{T'}(q) \\
\le \quad & a_{B_R(o_1)}(q)
\end{aligned}
$$

hence, in the two cases, $a_T(q) \leq a_R(q)$.

Lastly, let $q \in \mathcal{P}^f(T)$. By Fact 18, we can distinguish between the two following cases:

- there exists $q' \in \mathcal{P}^f(T') \setminus \mathcal{P}^f_\circ(B_R(o_1))$ such that $q = (o_1, e_1) : q'$: by Proposition 5, we have $q \in \mathcal{K}_{k,j}(\mathcal{T}(e)[i])$;

- $q \in \text{im}(b_R(o_1))$: since, for any $q' \in \mathcal{K}_{k,j}(\mathcal{T}(e_1)[i])$, we have $a_{\mathcal{T}(e_1)[i]}(q') \geq k$, we obtain $q \in \mathcal{P}^f_\circ(T')$, hence $q \in \mathcal{K}_{k,j}(\mathcal{T}(e_1)[i])$; by Proposition 5, we obtain $q \in \mathcal{K}_{k,j}(\mathcal{T}(e)[i])$.

$\square$

D.3 $\quad \bigcup_{j \in \mathcal{N}_i(e)} \mathcal{S}^k_{\mathcal{T}(e)[i+1]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i])) \subseteq \bigcup_{j \in \mathcal{N}_i(e)} \mathcal{S}^k_{\mathcal{T}(e)[i]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i]))$

We said also that we do not add new connected components (the new boxes are never caught by the elements of the $\mathcal{S}^k_{\mathcal{T}(e)[i+1]}(\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}(e)[i]))$). It is the content of Lemma 24.

Fact 21. *Let R be a $\circ$-PS. Let $k > \mathsf{Card}(\mathcal{B}(R)), cosize(R)$. Let e be a k-injective pseudo-experiment of R. Let $i \in \mathbb{N}$. Let $T \sqsubseteq \mathcal{T}(e)[i], \mathcal{T}(e)[i+1]$. Let $p, p' \in \mathcal{P}_0(T)$ such that $p \frown_{\mathcal{T}(e)[i]} p'$. Then we have $p \frown_{\mathcal{T}(e)[i+1]} p'$.*

Proof. We distinguish between four cases:

- $p \in \mathcal{W}_0(\mathcal{T}(e)[i])$ and $p' = t_{\mathcal{G}(\mathcal{T}(e)[i])}(p)$: we have $p' = t_{\mathcal{G}(\mathcal{T}(e)[i+1])}(p)$;

- $p' \in \mathcal{W}_0(\mathcal{T}(e)[i])$ and $p = t_{\mathcal{G}(\mathcal{T}(e)[i])}(p')$: we have $p = t_{\mathcal{G}(\mathcal{T}(e)[i+1])}(p')$;

- $\{p, p'\} \in \mathcal{A}(\mathcal{G}(\mathcal{T}(e)[i]))$: we have $\{p, p'\} \in \mathcal{A}(\mathcal{G}(\mathcal{T}(e)[i+1]))$;

- there exists $o \in \mathcal{B}_0^{<i}(\mathcal{T}(e)[i])$ such that $p, p' \in \text{im}(b_{\mathcal{T}(e)[i]}(o))$: by Fact 3, we obtain $p \frown_{\mathcal{T}(e)[i+1]} p'$.

$\square$

Fact 22. *Let R be a $\circ$-PS. Let $k > \mathsf{Card}(\mathcal{B}(R)), cosize(R)$. Let e be a k-injective pseudo-experiment of R. Let $i \in \mathbb{N}$. Let $T \sqsubseteq \mathcal{T}(e)[i], \mathcal{T}(e)[i+1]$. Let $p, p' \in \mathcal{P}_0(T)$ such that $p \frown_{\mathcal{T}(e)[i+1]} p'$. Then we have $p \frown_{\mathcal{T}(e)[i]} p'$ or $p, p' \in \mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}(e)[i])$.*

Proof. We distinguish between three cases:

- $(p \in \mathcal{W}_0(\mathcal{T}(e)[i+1])$ and $p' = t_{\mathcal{G}(\mathcal{T}(e)[i+1])}(p))$ or $(p' \in \mathcal{W}_0(\mathcal{T}(e)[i+1])$ and $p = t_{\mathcal{G}(\mathcal{T}(e)[i+1])}(p'))$ or $\{p, p'\} \in \mathcal{A}(\mathcal{G}(\mathcal{T}(e)[i+1]))$: we have $p \frown_{\mathcal{T}(e)[i]} p'$;

- there exists $o \in \mathcal{B}_0^{<i}(\mathcal{T}(e)[i+1])$ such that $p, p' \in \text{im}(b_{\mathcal{T}(e)[i+1]}(o))$: by Fact 3, we have $p \frown_{\mathcal{T}(e)[i]} p'$;

- there exists $o \in \mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1])$ such that $p, p' \in \text{im}(b_{\mathcal{T}(e)[i+1]}(o))$: by Proposition 5, we obtain $p, p' \in \mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}(e)[i])$.

$\square$

Fact 23. *Let R be a $\circ$-PS. Let $k > \mathsf{Card}(\mathcal{B}(R)), cosize(R)$. Let e be a k-injective pseudo-experiment of R. Let $i \in \mathbb{N}$. Let $j \in \mathcal{N}_i(e)$. Let $T \in \mathcal{S}^k_{\mathcal{T}(e)[i+1]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i]))$ such that $depth(T) \leq i$. Then we have $\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}(e)[i]) \cap \mathcal{P}_0(T) \subseteq \mathcal{K}_{k,j}(\mathcal{T}(e)[i])$.*

Proof. Let $p \in \mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}(e)[i]) \cap \mathcal{P}_0(T)$. By Proposition 5, there exists $o \in \mathcal{B}_0^{=i}(\mathcal{T}(e)[i+1])$ such that $p \in \text{im}(b_{\mathcal{T}(e)[i+1]}(o))$. We have $p \frown_{\mathcal{T}(e)[i+1]} o$, hence $p \notin \mathcal{K}_{k,j}(\mathcal{T}(e)[i]) \Rightarrow o \in \mathcal{P}_0(T)$. But, since $depth(T) \leq i$, we have $o \notin \mathcal{P}_0(T)$. $\square$

Lemma 24. *Let R be a $\circ$-PS. Let $k > \mathsf{Card}(\mathcal{B}(R)), cosize(R)$. Let e be a k-injective pseudo-experiment of R. Let $i \in \mathbb{N}$. For any $j \in \mathcal{N}_i(e)$, we have $\mathcal{S}^k_{\mathcal{T}(e)[i+1]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i])) \subseteq \mathcal{S}^k_{\mathcal{T}(e)[i]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i]))$.*

Proof. (Sketch) Let $j \in \mathcal{N}_i(e)$ and $T \in \mathcal{S}^k_{\mathcal{T}(e)[i+1]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i]))$. By assumption we have $\mathcal{P}_0(T) \backslash \mathcal{K}_{k,j}(\mathcal{T}(e)[i]) \neq \emptyset$.

First, notice that $depth(T) \leq i$. Indeed, assume that $depth(T) = i + 1$. Let $o \in \mathcal{B}_0^{=i}(T) = \mathcal{B}_0^{=i}(\mathcal{T}(e)[i + 1]) \cap \mathcal{P}_0(T)$. For any $p \in \mathcal{P}_0(T)$ such that $o \frown_T p$, we have $p \in \mathcal{P}^{\mathsf{f}}(T)$. Indeed, consider the case $o \in \mathcal{B}_0^{=i}(R)$: we have $b_{\mathcal{T}(e)[i+1]}(o) = b_R(o)$; by Proposition 5, we have $\{q \in \mathcal{P}_0(\mathcal{T}(e)[i]); q \in \mathsf{im}(b_R(o))\} = \mathsf{im}(b_R(o)) = \mathcal{K}_{k,j}(\mathcal{T}(e)[i])$. This shows that $\mathcal{P}_0(T) = \mathcal{P}^{\mathsf{f}}(T)$, which contradicts $T \in \mathcal{S}^k_{\mathcal{T}(e)[i+1]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i]))$.

Now, using Fact 3, we could easily show that $T \sqsubseteq \mathcal{T}(e)[i]$. So, by Fact 22 and Fact 23, we have $T \trianglelefteq_{\mathcal{K}_{k,j}(\mathcal{T}(e)[i])} \mathcal{T}(e)[i]$. Finally, by Fact 21, we obtain $T \in \mathcal{S}^k_{\mathcal{T}(e)[i]}(\mathcal{K}_{k,j}(\mathcal{T}(e)[i]))$. □

E   With axioms (Remark 5)

With axioms, we need to slightly modify Definition 7, since different experiments can induce the same pseudo-experiment:

Definition 22. *Given an experiment e of some differential $\circ$-PS R, we define, by induction on $depth(R)$, a pseudo-experiment $\overline{e}$ of R as follows: $\overline{e}(\varepsilon) = (R, 1)$ and*

$$\overline{e}(o) = \left\{ \overline{f}[\varepsilon \mapsto (B_R(o), i)]; f \in Supp(\mathcal{B}(e)(o)) \text{ and } 1 \leq i \leq \sum_{\substack{g \in Supp(\mathcal{B}(e)(o)) \\ \overline{g} = \overline{f}}} \mathcal{B}(e)(o)(g) \right\}$$

for any $o \in \mathcal{B}_0(R)$

Notice that, if there is no axiom, Definitions 7 and 22 induce the same pseudo-experiment $\overline{e}$ for an experiment e.

F   Untyped framework (Remark 6)

Since there is no type, we define (differential) ground-structures *via* the auxiliary definition of *(differential) pre-ground-structures*:

Definition 23. *A differential pre-ground-structure is a 6-tuple $\mathcal{G} = (\mathcal{W}, \mathcal{P}, l, t, \mathcal{L}, \mathcal{A})$, where*

- *$\mathcal{P}$ is a finite set; the elements of $\mathcal{P}(\mathcal{G})$ are the* ports *of $\mathcal{G}$;*

- *$\mathcal{W}$ is a subset of $\mathcal{P}$; the elements of $\mathcal{W}(\mathcal{G})$ are the* wires *of $\mathcal{G}$;*

- *l is a function $\mathcal{P} \to \mathfrak{T}$ such that $(\forall w \in \mathcal{W}) l(w) \neq \circ$; the element $l(p)$ of $\mathfrak{T}$ is the* label *of p in $\mathcal{G}$;*

- *t is a function $\mathcal{W} \to \mathcal{P}$ such that, for any $p \in \mathcal{P}$, we have*

   - *$l(p) \in \{\otimes, \mathfrak{N}\} \Rightarrow \mathsf{Card}(\{w \in \mathcal{W}; t(w) = p\}) = 2$;*
   - *$l(p) \in \{1, \bot, ax\} \Rightarrow \mathsf{Card}(\{w \in \mathcal{W}; t(w) = p\}) = 0$;*

   if $t(w) = p$, then w is a premise *of p;*

- *$\mathcal{L}$ is a subset of $\{w \in \mathcal{W}; l(t(w)) \in \{\otimes, \mathfrak{N}\}\}$ such that, for any $p \in \mathcal{P}$ such that $l(p) \in \{\otimes, \mathfrak{N}\}$, we have $\mathsf{Card}(\{w \in \mathcal{L}; t(w) = p\}) = 1$; if $w \in \mathcal{L}$ such that $t(w) = p$, we say that w is a* left premise *of p;*

- *and $\mathcal{A}$ is a partition of $\{p \in \mathcal{P}; l(p) = ax\}$ such that, for any $a \in \mathcal{A}$, $\mathsf{Card}(a) = 2$; the elements of $\mathcal{A}$ are the* axioms *of $\mathcal{G}$.*

We set $\mathcal{W}(\mathcal{G}) = \mathcal{W}$, $\mathcal{P}(\mathcal{G}) = \mathcal{P}$, $l_\mathcal{G} = l$, $t_\mathcal{G} = t$, $\mathcal{L}(\mathcal{G}) = \mathcal{L}$, $\mathcal{A}(\mathcal{G}) = \mathcal{A}$ and $\mathcal{P}^f(\mathcal{G}) = \mathcal{P} \setminus \mathcal{W}$. The elements of $\mathcal{P}^f(\mathcal{G})$ are the conclusions of $\mathcal{G}$.

We set $\mathcal{P}^1(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = 1\}$, $\mathcal{P}^\perp(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = \perp\}$, $\mathcal{P}^!(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = !\}$, $\mathcal{P}^?(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = ?\}$, $\mathcal{P}^\circ(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = \circ\}$, $\mathcal{P}^e(\mathcal{G}) = \mathcal{P}^!(\mathcal{G}) \cup \mathcal{P}^?(\mathcal{G}) \cup \mathcal{P}^\circ(\mathcal{G})$, $\mathcal{P}^\otimes(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = \otimes\}$, $\mathcal{P}^{\invamp}(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = \invamp\}$ and $\mathcal{P}^m(\mathcal{G}) = \mathcal{P}^\otimes(\mathcal{G}) \cup \mathcal{P}^{\invamp}(\mathcal{G})$.

A pre-ground-structure is a differential pre-ground-structure $\mathcal{G}$ such that $im(t_\mathcal{G}) \cap (\mathcal{P}^!(\mathcal{G}) \cup \mathcal{P}^\circ(\mathcal{G})) = \emptyset$.

A differential ground-structure (resp. a ground-structure) is a differential pre-ground structure (resp. a pre-ground structure) $\mathcal{G}$ such that the reflexive transitive closure $<_\mathcal{G}$ of the binary relation $<$ on $\mathcal{P}(\mathcal{G})$ defined by $p < p'$ iff $p = t_\mathcal{G}(p')$ is antisymmetric.

For the semantics of PS's, we are given a set A that does not contain any couple nor any 3-tuple and such that $* \notin A$. We define, by induction on n, the set $D_{A,n}$ for any $n \in \mathbb{N}$:

- $D_{A,0} = \{+, -\} \times (A \cup \{*\})$

- $D_{A,n+1} = D_{A,0} \cup (\{+, -\} \times D_{A,n} \times D_{A,n}) \cup (\{+, -\} \times \mathcal{M}_{fin}(D_{A,n}))$

We set $D_A = \bigcup_{n \in \mathbb{N}} D_{A,n}$.

Definition 25 is an adaptation of Definition 6 in an untyped framework.

Definition 24. *For any $\alpha \in D_A$, we define $\alpha^\perp \in D_A$ as follows:*

- *if $\alpha \in A$ and $\delta \in \{+, -\}$, then $(\delta, \alpha)^\perp = (\delta^\perp, \alpha)$;*

- *if $\alpha = (\delta, *)$ with $\delta \in \{+, -\}$, then $\alpha^\perp = (\delta^\perp, *)$;*

- *if $\alpha = (\delta, \alpha_1, \alpha_2)$ with $\delta \in \{+, -\}$ and $\alpha_1, \alpha_2 \in D_A$, then $\alpha^\perp = (\delta^\perp, \alpha_1^\perp, \alpha_2^\perp)$;*

- *if $\alpha = (\delta, [\alpha_1, \ldots, \alpha_m])$ with $\delta \in \{+, -\}$ and $\alpha_1, \ldots, \alpha_m \in D_A$, then $\alpha^\perp = (\delta^\perp, [\alpha_1^\perp, \ldots, \alpha_m^\perp])$;*

where $+^\perp = -$ and $-^\perp = +$.

Definition 25. *For any differential $\circ$-PS R, we define, by induction on $depth(R)$ the set of* experiments *of R: it is is the set of triples $(R, e_\mathcal{P}, e_\mathcal{B})$, where $e_\mathcal{P}$ is a function $\mathcal{P}_0(R) \to D_A \cup \mathcal{M}_{fin}(D_A)$ and $e_\mathcal{B}$ is a function which associates to every $o \in \mathcal{B}_0(R)$ a finite multiset of experiments of $B_R(o)$ such that*

- *for any $\{p, q\} \in \mathcal{A}(\mathcal{G}(R))$, we have $e_\mathcal{P}(p) = \alpha$, $e_\mathcal{P}(q) = \alpha^\perp$ for some $\alpha \in D_A$;*

- *for any $p \in \mathcal{P}_0^\otimes(R)$ (resp. $p \in \mathcal{P}_0^{\invamp}(R)$), for any $w_1, w_2 \in \mathcal{W}_0(R)$ such that $t_{\mathcal{G}(R)}(w_1) = p = t_{\mathcal{G}(R)}(w_2)$, $w_1 \in \mathcal{L}(\mathcal{G}(R))$ and $w_2 \notin \mathcal{L}(\mathcal{G}(R))$, we have $e_\mathcal{P}(p) = (+, e_\mathcal{P}(w_1), e_\mathcal{P}(w_2))$ (resp. $e_\mathcal{P}(p) = (-, e_\mathcal{P}(w_1), e_\mathcal{P}(w_2))$);*

- *for any $p \in \mathcal{P}_0^1(R)$ (resp. $p \in \mathcal{P}_0^\perp(R)$), we have $e_\mathcal{P}(p) = (+, *)$ (resp. $e_\mathcal{P}(p) = (-, *)$);*

- *for any $p \in \mathcal{P}_0^e(R)$, we have $e(p) = \begin{cases} a & \text{if } p \in \mathcal{P}_\circ^f(R); \\ (-, a) & \text{if } p \in \mathcal{P}_0^?(R); \quad \text{where} \\ (+, a) & \text{if } p \in \mathcal{P}_0^!(R); \end{cases}$*

$$a = \sum_{\substack{w \in \mathcal{W}_0(R) \\ t_{\mathcal{G}(R)}(w) = p}} [e_\mathcal{P}(w)] +$$

$$\sum_{o \in \mathcal{B}_0(R)} \sum_{e' \in Supp(e_\mathcal{B}(o))} \left(\sum_{\substack{q \in \mathcal{P}_\bullet^f(B_R(o)) \\ b_R(o)(q) = p}} e_\mathcal{B}(o)(e') \cdot [e'_\mathcal{P}(q)] + \sum_{\substack{q \in \mathcal{P}_\circ^f(B_R(o)) \\ b_R(o)(q) = p}} e_\mathcal{B}(o)(e') \cdot e'_\mathcal{P}(q) \right)$$

For any experiment $e = (R, e_{\mathcal{P}}, e_{\mathcal{B}})$, we set $\mathcal{P}(e) = e_{\mathcal{P}}$ and $\mathcal{B}(e) = e_{\mathcal{B}}$.

For any differential $\circ$-PS R, we set $[\![R]\!]_A = \{\mathcal{P}(e)|_{\mathcal{P}^f(R)}; e$ is an experiment of $R\}$.

Definition 26. *Let $r \in \mathcal{M}_{fin}(D_A)$. We say that r is* injective *if, for every $\gamma \in A$, there are at most two occurrences of γ in r.*

For any set $\mathcal{P}$, for any function $x : \mathcal{P} \to D_A$, we say that x is injective *if $\sum_{p \in \mathcal{P}}[x(p)]$ is injective.*

An experiment e of a differential $\circ$-PS S is said to be injective *if $\mathcal{P}(e)|_{\mathcal{P}^f(R)}$ is injective.*

Definition 27. *Let $\sigma : A \to D_A$. For any $\alpha \in D_A$, we define $\sigma \cdot \alpha \in D_A$ as follows:*

- *if $\alpha \in A \cup \{*\}$, then $\sigma \cdot (+, \alpha) = \sigma(\alpha)$ and $\sigma \cdot (-, \alpha) = \sigma(\alpha)^{\perp}$;*

- *if $\delta \in \{+, -\}$ and $\alpha_1, \alpha_2 \in D_A$, then $\sigma \cdot (\delta, \alpha_1, \alpha_2) = (\delta, \sigma \cdot \alpha_1, \sigma \cdot \alpha_2)$;*

- *if $\delta \in \{+, -\}$ and $\alpha_1, \ldots, \alpha_m \in D_A$, then $\sigma \cdot (\delta, [\alpha_1, \ldots, \alpha_m]) = (\delta, [\sigma \cdot \alpha_1, \ldots, \sigma \cdot \alpha_m])$.*

For any set $\mathcal{P}$, for any function $x : \mathcal{P} \to D_A$, we define a function $\sigma \cdot x : \mathcal{P} \to D_A$ by setting: $(\sigma \cdot x)(p) = \sigma \cdot x(p)$ for any $p \in \mathcal{P}$.

Remark 8. *For any functions $\sigma, \sigma' : A \to D_A$, for any function $x : \mathcal{P} \to D_A$, we have $\sigma \cdot (\sigma' \cdot x) = (\sigma \cdot \sigma') \cdot x$.*

Definition 28. *Let S be a differential $\circ$-PS. Let e be an experiment of S. Let $\sigma : A \to D_A$. We define, by induction of $depth(S)$, an experiment $\sigma \cdot e$ of S by setting*

- $\mathcal{P}(\sigma \cdot e) = \sigma \cdot \mathcal{P}(e)$

- $\mathcal{B}(\sigma \cdot e)(o) = \sum_{e_1 \in Supp(\mathcal{B}(e)(o_1))} \mathcal{B}(e)(o_1)(e_1) \cdot [\sigma \cdot e_1]$ *for any $o_1 \in \mathcal{B}_0(S)$.*

Since we deal with untyped proof-nets, we cannot assume that the proof-nets are η-expanded and that experiments label the axioms only by atoms. That is why we introduce the notion of *atomic experiment*:

Definition 29. *For any differential $\circ$-PS R, we define, by induction on $depth(R)$, the set of* atomic experiments *of R: it is the set of experiments e of R such that*

- *for any $\{p, q\} \in \mathcal{A}(\mathcal{G}(R))$, we have $\mathcal{P}(e)(p), \mathcal{P}(e)(q) \in \{+, -\} \times A$;*

- *and, for any $o_1 \in \mathcal{B}_0(R)$, the multiset $\mathcal{B}(e)(o_1)$ is a multiset of atomic experiments of $B_R(o_1)$.*

Fact 25. *Let R be a $\circ$-PS. Let e be an experiment of R. If e is atomic, then $\mathcal{T}(e)$ is atomic.*

Proof. By induction on $depth(R)$. $\square$

Definition 30. *Let $\mathcal{P}$ be a set. Let $\mathcal{D} \subseteq (D_A)^{\mathcal{P}}$. Let $x \in \mathcal{D}$, we say that x is $\mathcal{D}$-atomic if we have*

$$(\forall \sigma \in (D_A)^A)(\forall y \in \mathcal{D})(\sigma \cdot y = x \Rightarrow (\forall \gamma \in At(y))\sigma(\gamma) \in A)$$

where $At(y)$ is the set of atoms occurring in $\mathsf{im}(y)$.

We denote by $\mathcal{D}_{At}$ the subset of $\mathcal{D}$ consisting of the $\mathcal{D}$-atomic elements of $\mathcal{D}$.

For any PS R, any $[\![R]\!]_A$-atomic injective point is the result of some atomic experiment of R:

Fact 26. *Let R be a $\circ$-PS. Let $x \in ([\![R]\!]_A)_{At}$ injective. Then there exists an atomic experiment e of R such that $e_{|\mathcal{P}^f(R)} = x$.*

Proof. We prove, by induction on $(depth(R), \mathsf{Card}(\mathcal{P}_0(R)))$ lexicographically ordered, that, for any non-atomic injective experiment e' of R, there exist an experiment e of R, a function $\sigma : A \to D_A$ such that $\sigma \cdot e = e'$ and $\gamma \in At(e_{|\mathcal{P}^f(R)})$ such that $\sigma(\gamma) \notin A$. $\square$
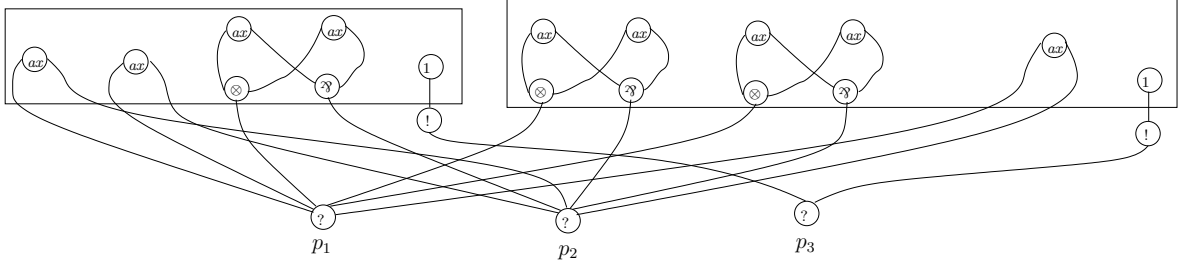
Figure 15: PS R''

The converse does not necessarily hold: for some PS's R, there are results of atomic injective experiments of R that are not $[\![R]\!]_A$-atomic. Indeed, consider Figure 15. There exists an atomic injective experiment e of R'' such that

- $\mathcal{P}(e)(p_1) = (-, [(+, \gamma_1), \ldots, (+, \gamma_7), (+, (+, \gamma_8), (+, \gamma_9)), \ldots, (+, (+, \gamma_{22}), (+, \gamma_{23}))])$,

- $\mathcal{P}(e)(p_2) = (-, [(-, \gamma_1), \ldots, (-, \gamma_7), (-, (-, \gamma_8), (-, \gamma_9)), \ldots, (-, (-, \gamma_{22}), (-, \gamma_{23}))])$

- and $\mathcal{P}(e)(p_3) = (-, [(+, [(+, *), (+, *)]), (+, [(+, *), (+, *), (+, *)])])$,

where $\{\gamma_1, \ldots, \gamma_{23}\} \subseteq A$. But $e_{|\{p_1, p_2, p_3\}}$ is not in $([\![R'']\!]_A)_{At}$: there exists an atomic injective experiment e' of R' such that

- $\mathcal{P}(e')(p_1) = (-, [(+, \gamma_1), \ldots, (+, \gamma_8), (+, (+, \gamma_{10}), (+, \gamma_{11})), \ldots, (+, (+, \gamma_{22}), (+, \gamma_{23}))])$,

- $\mathcal{P}(e')(p_2) = (-, [(-, \gamma_1), \ldots, (-, \gamma_8), (-, (-, \gamma_{10}), (-, \gamma_{11})), \ldots, (-, (-, \gamma_{22}), (-, \gamma_{23}))])$

- and $\mathcal{P}(e')(p_3) = (-, [(+, [(+, *), (+, *)]), (+, [(+, *), (+, *), (+, *)])])$;

we set $\sigma(\gamma) = \begin{cases} \gamma & \text{if } \gamma \in A \setminus \{\gamma_8\}; \\ (+, (+, \gamma_8), (+, \gamma_9)) & \text{if } \gamma = \gamma_8; \end{cases}$ - we have $\sigma \cdot e'_{|\{p_1, p_2, p_3\}} = e_{|\{p_1, p_2, p_3\}}$.

But it does not matter, because there are many enough atomic points:

Fact 27. *Let R be a $\circ$-PS. For any $y \in [\![R]\!]_A$, there exist $x \in ([\![R]\!]_A)_{At}$ and $\sigma : A \to D_A$ such that $\sigma \cdot x = y$.*

Proof. By induction on $size(\sum_{p \in \mathcal{P}^f(R)} [y(p)])$, where $size(r) \in \mathbb{N}$ is defined for any $r \in \mathcal{M}_{fin}(D_A)$ as follows: $size(r) = \sum_{\alpha \in Supp(r)} r(\alpha) \cdot \min\{i \in \mathbb{N}; \alpha \in D_{A,i}\}$: if $y \in ([\![R]\!]_A)_{At}$, then we can set $x = y$ and $\sigma = id_A$; if $y \notin ([\![R]\!]_A)_{At}$, then there exist a function $\sigma' : A \to D_A$, $y' \in [\![R]\!]_A$ such that $\sigma' \cdot y' = y$ and $\gamma \in At(y')$ such that $\sigma'(\gamma) \notin A$, hence $size(\sum_{p \in \mathcal{P}^f(R)} [y'(p)]) < size(\sum_{p \in \mathcal{P}^f(R)} [y(p)])$. By induction hypothesis, there exist $x \in ([\![R]\!]_A)_{At}$ and $\sigma'' : A \to D_A$ such that $\sigma'' \cdot x = y'$. We set $\sigma = \sigma' \cdot \sigma''$: we have $\sigma \cdot x = (\sigma' \cdot \sigma'') \cdot x = \sigma' \cdot (\sigma'' \cdot x) = \sigma' \cdot y' = y$. $\qquad \square$